On the Instability of Millionshchikov Linear Systems with Smooth Dependence on a Parameter

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Consider a one-parameter family of two-dimensional linear differential systems

$$\dot{x} = A_\mu(t)x, \;\; x \in \mathbb{R}^2, \;\; t \geq 0 \tag{1_μ}$$

with the matrices

$$A_\mu(t) := \begin{cases} d_k(\mu)\operatorname{diag}[1,-1], & 2k-2 \leq t < 2k-1, \\ (\mu + \gamma(\mu) + b_k)J, & 2k-1 \leq t < 2k, \end{cases}$$

where $k \in \mathbb{N}$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and a real parameter μ; the conditions on the numbers $b_k \in \mathbb{R}$ and the functions $d_k(\,\cdot\,), \gamma(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$ will be indicated below.

It was proved in [2] that the upper Lyapunov exponent of system (1_μ) considered as a function of the parameter μ is positive on a set of positive Lebesgue measure for the case in which the functions $d_k(\,\cdot\,)$ are independent of μ, positive, and separated from zero uniformly in $k \in \mathbb{N}$ (i.e., $d_k(\mu) \equiv d_k \geqslant d > 0$, $k \in \mathbb{N}$). Complex matrices of a special kind are substantially used in the proof of this result. Another method for proving the theorem in [1] based on an application of the Parseval equality for trigonometric sums can be found in [3].

Let $\alpha_n \in \mathbb{R}$, $n \in \mathbb{N}$ be arbitrary numbers. Set

$$d_k(\mu) \equiv d(\mu) > 0, \;\; b_{2^{n-1}} = \alpha_n, \;\; k \in \mathbb{N}, \;\; \mu \in \mathbb{R}. \tag{2}$$

Denote the Cauchy matrix of system (1_μ) by $X_{A_\mu}(t,s)$, $t, s \geq 0$. For each $\varphi \in \mathbb{R}$, the matrix of clockwise rotation by the angle φ will be denoted by

$$U(\varphi) \equiv \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

One can readily verify that if the matrix $A_\mu(\,\cdot\,)$ is determined by conditions (2), then

$$X_{A_\mu}(2^{k+1}, 0) = U(\alpha_{k+1} - \alpha_k) X^2_{A_\mu}(2^k, 0) \;\text{ for any }\; k \in \mathbb{N}.$$

Systems with coefficients chosen according to (2) have a number of properties that permit one to construct one-parameter families with various asymptotic characteristics. In particular, if the sequence $\{\alpha_n\}_{n=1}^\infty$ converges, then the matrix $A_\mu(\,\cdot\,)$ is the limit of a sequence of periodic matrices uniformly with respect to $t \geq 0$. V. M. Millionshchikov used such systems in [5–7] (see also [1]) to prove the existence of Lyapunov improper linear differential systems with limit-periodic and quasiperiodic coefficients.

In the paper [4], it was proved under conditions (2) in which $\gamma(\,\cdot\,) \equiv 0$ and in the case of a continuous function $d(\,\cdot\,)$ that there exists a parameter value $\mu \in \mathbb{R}$ such that system (1_μ) is unstable. In the present talk we show that the upper Lyapunov exponent of system (1_μ) considered as a function of the parameter μ is positive on a set of positive Lebesgue measure for the case in which the functions $d_k(\,\cdot\,)$ and $\gamma(\,\cdot\,)$ are differentiable and under the conditions

$$\widetilde{C} := \inf_{\mu\in\mathbb{R}} (1+\gamma'(\mu)) > 2|d'(\mu)|e^{4d(\mu)}, \ \ \mu\in\mathbb{R}, \tag{3}$$

$$\int\limits_0^\pi d(\mu)\,d\mu > 2^{10}(1+\widetilde{C}^{-1}). \tag{4}$$

For any $k\in\mathbb{N}$ and $\mu\in\mathbb{R}$ we recursively define real numbers $\eta_k=\eta_k(\mu)\geq 1$ and $\psi_k=\psi_k(\mu)$ as follows. Set

$$\eta_1(\mu)=e^{d(\mu)}, \quad \psi_1(\mu):\equiv 0,$$
$$\xi_k=\xi_k(\mu):=2\psi_k(\mu)+\alpha_k+\mu+\gamma(\mu), \quad q_k(\mu):=2\pi\big[2^{-1}\pi^{-1}\xi_k(\mu)\big]$$

(here $[\,\cdot\,]$ denotes the integer part of number). Since $\eta_k\geq 1$ and hence $\operatorname{sh}(2\ln\eta_k)\geq 0$, it follows that there exist unique $1\leq\eta_{k+1}\in\mathbb{R}$ and $\varphi_k=\varphi_k(\mu)\in[q_k(\mu)-2^{-1}\pi, q_k(\mu)+2^{-1}\pi)$ such that

$$\operatorname{sh}\ln\eta_{k+1}=(\operatorname{sh}(2\ln\eta_k))|\cos\xi_k|,$$
$$\operatorname{ctg}\varphi_k=(\operatorname{ch}(2\ln\eta_k))\operatorname{ctg}\xi_k \ \text{ if } \ \sin\xi_k\neq,$$
$$\varphi_k=\xi_k \ \text{ if } \ \sin\xi_k=0.$$

Finally, we set

$$\psi_{k+1}(\mu):=\psi_k(\mu)+2^{-1}\varphi_k(\mu)+\frac{\pi}{2}\,\beta(\mu),$$

where

$$\beta(\mu)=0 \ \text{ if } \ \xi_k(\mu)\in\bigcup_{n\in\mathbb{Z}}[2\pi n-2^{-1}\pi, 2\pi n+2^{-1}\pi),$$

$\beta(\mu)=1$ for all others $\mu\in\mathbb{R}$.

In what follows, we will assume that conditions (2) and (3) hold.

Lemma 1. *For any $n\in\mathbb{N}$ and $\mu\in\mathbb{R}$ the functions η_k and ψ_k are differentiable on μ and one has the representation*

$$X_{A_\mu}(2^n-1,0)=U(\psi_n)\begin{pmatrix}\eta_n & 0\\ 0 & \eta_n^{-1}\end{pmatrix}U(\psi_n).$$

Lemma 2. *For any $k\in\mathbb{N}$ an equality holds*

$$\psi_k(\pi)-\psi_k(0)=(2^{k-1}-2^{-1})\pi.$$

Besides of that for all $\mu\in\mathbb{R}$ we have the estimation

$$\psi_k'(\mu)>0.$$

Lemma 3. *For any $k\in\mathbb{N}$ the inequality is true*

$$\int\limits_0^\pi \ln|\cos\xi_k(\mu)|\,d\mu\geq -2^5k-2\pi\ln(1+\widetilde{C}^{-1}).$$

Theorem. *If conditions (2)–(4) are satisfied, then there exists a set $J\subset\mathbb{R}$ of positive Lebesgue measure such that the upper Lyapunov exponent $\lambda_2(A_\mu)$ of system (1_μ) considered as a function of the parameter μ is positive for all $\mu\in J$.*

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