Blow-Up Solutions of the Cauchy Problem for Nonlinear Delay Ordinary Differential Equations

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Problems on the existence and asymptotic estimates of blow-up solutions occupy an important place in the qualitative theory of ordinary differential equations and have been studied in sufficient detail for a wide class of nonlinear nonautonomous ordinary differential equations (see [1–9] and the references therein). However, for delay differential equations this problem remained practically unstudied. Most probably, [10] is the first work done in this direction. Here theorems on the existence of blow-up solutions are proved for the equation that does not contain intermediate derivatives. In the present paper, similar results are given for the equation of general type.

On a finite interval $[0,b[$ we investigate the delay differential equation

$$u^{(n)}(t)=f\big(t,u(\tau(t)),\dots,u^{(n-1)}(\tau(t))\big) \tag{1}$$

with the initial conditions

$$u^{(i-1)}(t)=c_i(t) \;\text{ for }\; a\le t\le 0 \;\; (i=1,\dots,n). \tag{2}$$

Here n is an arbitrary natural number, $f:[0,b]\times\mathbb{R}_+^n\to\mathbb{R}_+$ is a continuous function, $\mathbb{R}_+=[0,+\infty[$, $\tau:[0,b]\to\mathbb{R}$ is a continuous function, satisfying the conditions

$$\begin{gathered}\tau(t)<t \;\text{ for }\; 0\le t<b, \quad \tau(b)=b,\\ a=\min\big\{\tau(t):\; 0\le t\le b\big\},\end{gathered} \tag{3}$$

and $c_i:[a,0]\to\mathbb{R}_+$ $(i=1,\dots,n)$ are also continuous functions.

Definition 1. Let $t_0\in[0,b[$ and

$$t_*=\min\big\{\tau(t):\; t_0\le t\le b\big\}.$$

An n-times continuously differentiable function $u:[t_0,b[\,\to\mathbb{R}_+$ is said to be **a solution of equation** (1) **in the interval** $[t_0,b[$ if

$$u^{(i-1)}(t)\ge 0 \;\text{ for }\; t_0\le t<b \;\; (i=1,\dots,n),$$

and there exist continuous functions $u_{0i}:[t_*,t_0]\to\mathbb{R}_+$ $(i=1,\dots,n)$ such that in that interval equality (1) is satisfied, where

$$u^{(i-1)}(t)=u_{0i}(t) \;\text{ for }\; t_*\le t\le t_0 \;\; (i=1,\dots,n).$$

A solution u of equation (1), defined in the interval $[0,b[$ and satisfying the initial conditions (2), is said to be **a solution of problem** (1), (2).

Definition 2. A solution u of equation (1) defined in some interval $[t_0,b[$ is said to be **blow-up** if

$$\lim_{t\to b} u^{(n-1)}(t)=+\infty.$$

A blow-up solution $u:[t_0,b[\,\to\mathbb{R}_+$ is said to be **strongly blow-up (weakly blow-up)** if

$$\lim_{t\to b} u(t)=+\infty \quad \Big(\lim_{t\to b} u(t)<+\infty\Big).$$

Definition 3. A solution $u:[t_0,b[\,\to\mathbb{R}_+$ of equation (1), having the finite limits $\lim\limits_{t\to b} u^{(i-1)}(t)$ $(i=1,\dots,n)$, is said to be **regular**.

According to condition (3), there exists an increasing sequence of numbers $t_i\in\,]0,b[$ $(i=1,2,\dots)$ such that

$$\tau(t)<0 \ \text{ for } \ 0\le t<t_1, \quad \tau(t_1)=0,$$
$$\tau(t)<t_i \ \text{ for } \ t_i\le t<t_{i+1}, \quad \tau(t_{i+1})=t_i \ \ (i=1,2,\dots),$$
$$\lim_{i\to+\infty} t_i=b.$$

From this fact it immediately follows

Lemma 1. *For arbitrarily fixed continuous functions $c_i:[a,0]\to\mathbb{R}_+$ $(i=1,2,\dots,n)$, problem (1), (2) in the interval $[0,b[$ has a unique solution u and for any natural number k the equality*

$$u(t)=u_k(t) \ \ for \ \ 0\le t\le t_k$$

is valid, where

$$u_1^{(i-1)}(t)=c_i(t) \ \ for \ \ a\le t\le 0 \ \ (i=1,\dots,n), \quad u_1(t)=\sum_{i=1}^n \frac{c_i(0)}{(i-1)!}\,t^{i-1}$$
$$+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f\big(s,c_1(\tau(s)),\dots,c_n(\tau(s))\big)\,ds \ \ for \ \ 0\le t\le t_1,$$

$$u_{k+1}^{(i-1)}(t)=c_i(t) \ \ for \ \ a\le t\le 0 \ \ (i=1,\dots,n), \quad u_{k+1}(t)=\sum_{i=1}^n \frac{c_i(0)}{(i-1)!}\,t^{i-1}$$
$$+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f\big(s,u_k(\tau(s)),\dots,u_k^{(n-1)}(\tau(s))\big)\,ds \ \ for \ \ 0\le t\le t_{k+1} \ \ (k=1,2,\dots).$$

Theorem 1. *Let along with (3) the condition*

$$f(t,x_1,\dots,x_n)\ge f_0(t,x_1,\dots,x_n) \ \ for \ \ t_0\le t\le b, \ \ (x_1,\dots,x_n)\in\mathbb{R}_+^n$$

be satisfied, where $t_0\in\,]0,b[$, and $f_0:[0,b]\times\mathbb{R}_+^n\to\mathbb{R}_+$ is a nondecreasing in the phase variables continuous function such that the differential equation

$$v^{(n)}(t)=f_0\big(t,v(\tau(t)),\dots,v^{(n-1)}(\tau(t))\big)$$

in the interval $[t_0,b[$ has a blow-up solution v. Then there exist numbers $r>0$ and $t^*\in\,]t_0,b]$ such that if

$$c_n(0)>r, \tag{4}$$

then the solution u of problem (1), (2) is blow-up as well and admits the estimates

$$u^{(i-1)}(t)\ge v^{(i-1)}(t) \ \ for \ \ t^*\le t<b \ \ (i=1,\dots,n).$$

Based on this comparison theorem, effective criteria for the existence of blow-up solutions of problem (1), (2) are obtained. In particular, the following statement is true.

Corollary 1. *Let the functions f and τ satisfy the inequalities*

$$f(t,x_1,\dots,x_n)\geq \ell(b-t)^{\mu}x_k^{\lambda} \ \text{ for } \ t_0\leq t\leq b, \ \ (x_1,\dots,x_n)\in\mathbb{R}_+^n,$$
$$\alpha(t-b)+b\leq\tau(t)<t \ \text{ for } \ 0\leq t<b,$$

where $k\in\{1,\dots,n\}$, $t_0\in\,]0,b[\,$, $\ell>0$, $\mu\geq 0$, $\lambda>1$, $\alpha>1$. Then for an arbitrary $\gamma>0$ there exists a positive number $r=r(\gamma)$ such that if inequality (4) *holds, then the solution u of problem* (1), (2) *is strongly blow-up and admits the estimate*

$$\inf\Big\{(b-t)^{\gamma}u(t):\ t_0\leq t<b\Big\}>0. \tag{5}$$

An important particular case of (1) is the differential equation

$$u^{(n)}(t)=\sum_{i=1}^{n-1}p_i(t)\big(u^{(i-1)}(\alpha(t-b)+b)\big)^{\lambda_i}, \tag{6}$$

where $p_i:\mathbb{R}_+\to\mathbb{R}_+$ $(i=1,\dots,n)$ are continuous functions, $\lambda>1$, $\alpha>1$.

For this equation we consider the Cauchy problem with the initial conditions (2), where $a=-(\alpha-1)b$, and $c_i:[a,0]\to\mathbb{R}_+$ $(i=1,\dots,n)$ are continuous functions.

Corollary 2. *There exists $\varepsilon>0$ such that if*

$$\sum_{i=1}^{n}c_i(t)<\varepsilon \ \text{ for } \ a\leq t\leq 0,$$

then the solution of problem (6), (2) *is regular. And if*

$$p_k(t)\geq \ell(b-t)^{\mu} \ \text{ for } \ 0\leq t\leq b,$$

where $k\in\{1,\dots,n\}$, $\ell>0$, $\mu\geq 0$, then for an arbitrary $\gamma>0$ there exists a positive number $r=r(\gamma)$ such that in the case where inequality (4) *holds, the solution u of problem* (6), (2) *is strongly blow-up and admits estimate* (5).

The first part of the corollary can be easily obtained from Lemma 1, while the second part follows from Corollary 1.

Example 1. Let $n>2$, $\alpha>1$, $\lambda>2$, $\ell_0=((\lambda-1)\alpha^{\frac{\lambda}{1-\alpha}})^{\frac{1}{1-\lambda}}$, $b>0$, $a=-(\alpha-1)b$. We choose positive numbers ρ_i $(i=1,\dots,n)$ so that the function, defined by the equality

$$u(t)=\sum_{i=1}^{n-1}\frac{(t-a)^{i-1}}{(i-1)!}\,\rho_i+(-1)^{n-1}\ell_0\prod_{i=1}^{n-1}\Big(n-i-\frac{1}{\lambda-1}\Big)^{-1}(b-t)^{n-1-\frac{1}{\lambda-1}} \ \text{ for } \ a\leq t<b,$$

satisfies the conditions

$$u^{(i-1)}(t)\geq 0 \ \text{ for } \ a\leq t\leq 0 \ \ (i=1,\dots,n-1).$$

Then the restriction of the function u to $[0,b[$ is a solution of the differential equation

$$u^{(n)}(t)=\big(u^{(n-1)}(\alpha(t-b)+b)\big)^{\lambda} \tag{7}$$

with the initial functions

$$c_i(t) = u^{(i-1)}(t) \ \text{ for } \ a \le t \le 0 \ \ (i = 1, \dots, n).$$

Moreover, it is clear that $u^{(i-1)}$ $(i = 1, \dots, n-1)$ have finite limits

$$u^{(i-1)}(b-0) \ \ (i = 1, \dots, n-1),$$

and

$$\lim_{t \to b} u^{(n-1)}(t) = +\infty.$$

Consequently, u is a weakly blow-up solution of equation (6).

On the other hand, by virtue of Corollary 2 equation (7) has infinite sets of strongly blow-up and regular solutions.

The example constructed above shows that if the functions f and τ satisfy the conditions of either Theorem 1 or one of its corollaries, then equation (7) can simultaneously have strongly blow-up, weakly blow-up and regular solutions.

Example 2. Theorem 1 and its corollaries are specific for delay equations and they have no analogs for equations without delay. To make sure of this, in the interval $[0, b[$ we consider the differential equation

$$u^{(n)}(t) = (u^{(n-1)}(t))^{\lambda}, \tag{8}$$

where $n \ge 2$, $\lambda > 2$. We choose positive numbers ρ_i $(i = 1, \dots, n)$ so that the function, defined by the equality

$$u_0(t) = \sum_{i=1}^{n-1} \frac{\rho_i}{(i-1)!}\, t^{i-1} + (-1)^{n-1}(\lambda - 1)^{\frac{1}{1-\lambda}} \prod_{i=1}^{n-1} \Big(n - i - \frac{1}{\lambda - 1}\Big)^{-1} (b-t)^{n-1-\frac{1}{1-\lambda}} \ \text{ for } \ 0 \le t < b,$$

satisfies the conditions

$$u_0^{(i-1)}(0) \ge 0 \ \ (i = 1, \dots, n-1).$$

Then every solution of equation (8), defined in the interval $[0, b[$ and blowing up at the point b, has the form $u(t) \equiv u_0(t)$, and, consequently, it is weakly blow-up. On the other hand, no matter how the number

$$r > u_0^{(n-1)}(0)$$

is, equation (8) does not have a solution $u : [0, b[\to \mathbb{R}_+$, satisfying the inequalities

$$u^{(i-1)}(0) \ge 0 \ \ (i = 1, \dots, n-1), \quad u^{(n-1)}(0) \ge r.$$

Theorem 2. *Let $n \ge 2$,*

$$\tau(t) = \frac{b(t - t_0)}{b - t_0} \ \text{ for } \ 0 \le t \le b,$$

and let the function f satisfy the inequality

$$f(t, x_1, \dots, x_n) \ge \ell (b-t)^{\mu} \omega(x_k) \ \text{ for } \ t_0 \le t \le b, \ \ (x_1, \dots, x_n) \in \mathbb{R}_+^n,$$

where $t_0 \in]0, b[$, $k \in \{1, \dots, n\}$, $\ell > 0$, $\mu \ge 0$, and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Let, moreover, there exist a number $\lambda > 1$ such that

$$\int_0^x \omega(y)\, dy > x^{\lambda} - 1 \ \text{ for } \ x \ge 0. \tag{9}$$

Then for an arbitrary $\gamma > 0$ there exists a positive number $r = r(\gamma)$ such that if inequality (4) holds, then the solution u of problem (1), (2) is strongly blow-up and admits estimate (5).

Example 3. Consider the differential equation

$$u^{(n)}(t)=(b-y)^{\mu}\omega\Big(u\Big(\frac{b(t-t_0)}{b-t_0}\Big)\Big), \tag{10}$$

where $\mu\geq 0$, $t_0\in\,]a,b[\,$, and $\omega:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function which along with (9) satisfies the condition

$$\omega(x_m)=0 \ \ (m=1,2,\dots). \tag{11}$$

Here $\lambda>1$, and $x_m\in\mathbb{R}_+$ $(m=1,2,\dots)$ is an increasing sequence of numbers converging to $+\infty$. The example of such a function is constructed in [10, p. 44].

In view of (11), Theorem 1 and their corollaries leave open the question on the existence of blow-up solutions of equation (10). On the other hand, by Theorem 2 this equation has an infinite set of blow-up solutions.

References

[1] I. V. Astashova, On Kiguradze's problem on power-law asymptotic behavior of blow-up solutions to Emden–Fowler type differential equations. *Georgian Math. J.* **24** (2017), no. 2, 185–191.

[2] I. V. Astashova, Asymptotic behavior of singular solutions of Emden–Fowler type equations. (Russian) *Differ. Uravn.* **55** (2019), no. 5, 597–606; translation in *Differ. Equ.* **55** (2019), no. 5, 581–590.

[3] I. V. Astashova and M. Yu. Vasilev, On nonpower-law behavior of blow-up solutions to Emden–Fowler type higher-order differential equations. Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2018, Tbilisi, Georgia, December 1–3, pp. 11–15; https://rmi.tsu.ge/eng/QUALITDE-2018/Astashova_Vasilev_workshop_2018.pdf.

[4] R. Bellman, *Stability Theory of Differential Equations.* McGraw-Hill Book Co., Inc., New York–Toronto–London, 1953.

[5] N. A. Izobov, Continuable and noncontinuable solutions of a nonlinear differential equation of arbitrary order. (Russian) *Mat. Zametki* **35** (1984), no. 6, 829–839.

[6] I. T. Kiguradze, The asymptotic behavior of the solutions of a nonlinear differential equation of Emden–Fowler type. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 965–986.

[7] I. T. Kiguradze, *Some Singular Boundary Value Problems for Ordinary Differential Equations.* (Russian) Izdat. Tbilis. Univ., Tbilisi, 1975.

[8] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* Springer-Science, Business Media, B.V., 2012.

[9] I. T. Kiguradze and G. G. Kvinikadze, On strongly increasing solutions of nonlinear ordinary differential equations. *Ann. Mat. Pura Appl. (4)* **130** (1982), 67–87.

[10] I. Kiguradze and N. Partsvania, Rapidly growing and blow-up solutions to higher order nonlinear delay ordinary differential equations. *Mem. Differential Equations Math. Phys.* **90** (2023), 39–54.