Nonlocal Boundary Value Problems for Second Order Nonlinear Hyperbolic Systems

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In the rectangle Ω consider the boundary value problem

$$u_{xy} = f(x, y, u_x, u_y, u), \tag{1}$$

$$\ell\big(u(\,\cdot\,, y)\big) = \varphi(y), \quad h\big(u_x(x, \,\cdot\,)\big) = \psi(x), \tag{2}$$

where $\varphi \in C^1([0, \omega_2]; \mathbb{R}^n)$, $\psi \in C([0, \omega_1]; \mathbb{R}^n)$, $\ell : C([0, \omega_1]; \mathbb{R}^n) \to \mathbb{R}^n$ and $h : C([0, \omega_2]; \mathbb{R}^n) \to \mathbb{R}^n$ are bounded linear operators that are *commutative*, i.e., the operators ℓ and h satisfy the equality

$$\ell \circ h(z) = h \circ \ell(z) \quad \text{for} \quad z \in C(\Omega; \mathbb{R}^n).$$

By $\mathbf{B}^1(z; r)$ denote the closed ball of radius r centered at z in space $C^1(\Omega; \mathbb{R}^n)$, i.e.,

$$\mathbf{B}^1(z; r) = \big\{\zeta \in C^1(\Omega) : \ \|\zeta - z\|_{C^1(\Omega)} \leq r\big\}.$$

If $f(x, y, v, w, z)$ is differentiable with respect to the phase variables, set:

$$F_1(x, y, v, w, z) = \frac{\partial f(x, y, v, w, z)}{\partial v}\,, \quad F_2(x, y, v, w, z) = \frac{\partial f(x, y, v, w, z)}{\partial w}\,,$$

$$F_0(x, y, v, w, z) = \frac{\partial f(x, y, v, w, z)}{\partial z}\,,$$

$$P_j[u](x, y) = F_j\big(x, y, u_x(x, y), u_y(x, y), u(x, y)\big) \ \ (j = 0, 1, 2).$$

A vector function $(\widetilde{f}, \widetilde{\varphi}, \widetilde{\psi})$ s called an *admissible perturbation* if $\widetilde{f} \in C(\Omega \times \mathbb{R}^{3n}; \mathbb{R}^n)$ is locally Lipschitz continuous with respect to the first $2n$ phase variables, $\widetilde{\varphi} \in C^1([0, \omega_2]; \mathbb{R}^n)$, $\widetilde{\psi} \in C([0, \omega_1]; \mathbb{R}^n)$. Set: $\widetilde{F}_1(x, y, v, w, z) = \widetilde{f}_v(x, y, v, w, z)$ and $\widetilde{F}_2(x, y, v, w, z) = \widetilde{f}_w(x, y, v, w, z)$.

Definition 1. Let u_0 be a solution of problem (1), (2), and $r > 0$. Problem (1), (2) is said to be (u_0, r)-*well-posed* if:

(i) $u_0(x, y)$ is the unique solution of the problem in the ball $\mathbf{B}^1(u_0; r)$;

(ii) There exist a positive constant δ_0 and an increasing continuous function $\varepsilon : [0, \delta_0] \to [0, +\infty)$ such that $\varepsilon(0) = 0$ and for any $\delta \in (0, \delta_0]$ and an arbitrary admissible perturbation $(\widetilde{f}, \widetilde{\varphi}, \widetilde{\psi})$ satisfying the following conditions

$$\|\widetilde{F}_1(x, y, v, w, z)\| + \|\widetilde{F}_2(x, y, v, w, z)\| \leq \delta_0 \ \text{ for } \ (x, y, v, w, z) \in \Omega \times \mathbb{R}^{3n}, \tag{3}$$

$$\begin{gathered} \|\widetilde{f}(x, y, v, w, z)\| < \delta \ \text{ for } \ (x, y, v, w, z) \in \Omega \times \mathbb{R}^{3n}, \\ \|\widetilde{\varphi}\|_{C^1([0,\omega_2])} + \|\widetilde{\psi}\|_{C([0,\omega_1])} \leq \delta, \end{gathered} \tag{4}$$

the problem

$$u_{xy} = f(x, y, u_x, u_y, u) + \widetilde{f}(x, y, u_x, u_y, u), \tag{$\widetilde{1}$}$$

$$\ell\big(u(\,\cdot\,, y)\big) = \varphi(y) + \widetilde{\varphi}(y), \quad h\big(u_x(x, \,\cdot\,)\big) = \psi(x) + \widetilde{\psi}(x) \tag{$\widetilde{2}$}$$

has at least one solution in the ball $\mathbf{B}^1(u_0; r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0; \varepsilon(\delta))$.

Definition 2. Let u_0 be a solution of problem (1), (2), and $r > 0$. Problem (1), (2) is said to be *strongly* (u_0, r)*-well-posed* if:

(i) Problem (1), (2) is (u_0, r)-well-posed;

(ii) There exist positive numbers M_0 and δ_0 such that for arbitrary $\delta \in (0, \delta_0)$ an arbitrary admissible perturbation $(\widetilde{f}, \widetilde{\varphi}, \widetilde{\psi})$ satisfying inequalities (3), (4), problem ($\widetilde{1}$), ($\widetilde{2}$) has at least one solution in the ball $\mathbf{B}^1(u_0; r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0; M_0\,\delta)$.

Definition 3. Problem (1), (2) is called well-posed if it is (u_0, r)-well-posed for every $r > 0$.

Definition 4. A solution u_0 of problem (1), (2) is called *strongly isolated*, if problem (1), (2) is strongly (u_0, r)-well-posed for some $r > 0$.

The concepts of strong well-posedness and a strongly isolated solution of a boundary value problem for a nonlinear ordinary differential system were introduced in [1]. Definitions 2 and 4 are adaptations of the idea of Definitions 3.1 and 3.2 from [1] to problem (1), (2).

The linear case of system (1), i.e. the system

$$u_{xy} = P_1(x, y)u_x + P_2(x, y)u_y + P_0(x, y)u + q(x, y) \tag{5}$$

was studied in [2].

Along with problem (5), (2) consider its corresponding homogeneous problem

$$u_{xy} = P_1(x, y)u_x + P_2(x, y)u_y + P_0(x, y)u, \tag{5_0}$$

$$\ell(u(\,\cdot\,, y)) = 0, \quad h(u_x(x, \,\cdot\,)) = 0. \tag{2_0}$$

Definition 5. Problem (5), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi \in C^1([0, \omega_2]; \mathbb{R}^n)$, $\psi \in C([0, \omega_1]; \mathbb{R}^n)$ and $q \in C(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])} + \|q\|_{C(\Omega)}\Big),$$

where M is a positive constant independent of φ, ψ and q.

Remark 1. Notice that for the linear problem (5), (2) (u_0, r)-well-posedness is equivalent to the strong well-posedness. Furthermore, for problem (5), (2), Definitions 1, 2 and 3 are equivalent to Definition 5.

Theorem 1. *Let f be a continuously differentiable function with respect to the phase variables v, w and z, and let problem (1), (2) be strongly (u_0, r)-well-posed for some $r > 0$. Then problem (5_0), (2_0) is well-posed, where $P_j(x, y) = P_j[u_0](x, y)$ $(j = 0, 1, 2)$.*

Theorem 2. *Let f be a continuously differentiable function with respect to the phase variables v, w and z, and let there exist matrix functions $P_{ij} \in C(\Omega; \mathbb{R}^{n\times n})$ $(i = 1, 2;\ j = 0, 1, 2)$ such that*

(A_0)
$$P_{1j}(x,y) \le F_j(x,y,v,w,z) \le P_{2j}(x,y) \;\; for \;\; (x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n} \;\; (j=0,1,2,);$$

(A_1) *for every $x^* \in [0,\omega_1]$ and arbitrary measurable matrix function $P_1 : [0,\omega_2] \to \mathbb{R}^{n\times n}$ satisfying the inequalities*
$$P_{11}(x^*,y) \le P_1(y) \le P_{21}(x^*,y) \;\; for \;\; y \in [0,\omega_2],$$
the homogeneous problem
$$v' = P_1(y)v, \quad h(v) = 0$$
has only the trivial solution;

(A_2) *for every $y^* \in [0,\omega_2]$ and arbitrary measurable matrix function $P_2 : [0,\omega_1] \to \mathbb{R}^{n\times n}$ satisfying the inequalities*
$$P_{12}(x,y^*) \le P_2(x) \le P_{22}(x,y^*) \;\; for \;\; x \in [0,\omega_1],$$
the homogeneous problem
$$v' = P_2(x)v, \quad \ell(v) = 0$$
has only the trivial solution;

(A_3) *for arbitrary measurable matrix function $P_j : \Omega \to \mathbb{R}^{n\times n}$ $(j=0,1,2)$ satisfying the inequalities*
$$P_{1j}(x,y) \le P_j(x,y) \le P_{2j}(x,y) \;\; for \;\; (x,y) \in \Omega \;\; (j=0,1,2),$$
problem $(5_0),(2_0)$ has only the trivial solution.

Then problem $(1),(2)$ is ***strongly*** *well-posed.*

Remark 2. Conditions (A_1) and (A_2) of Theorem 2 are key and cannot be weakened. Violation of either of conditions (A_1) and (A_2) may lead to additional compatibility conditions between the boundary values (2) and the right-hand side of system (1).

Indeed, consider the problem
$$u_{xy} = P_2\, u_y + q(x,y,u), \tag{6}$$
$$u(0,y) = \varphi(y), \quad u_x(x,0) - u_x(x,\omega_2) = 0, \tag{7}$$
where $P_2 \in \mathbb{R}^{n\times n}$ is an arbitrary matrix, and $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ and $q \in C(\Omega\times\mathbb{R};\mathbb{R}^n)$ satisfy the equalities
$$\varphi(0) = \varphi(\omega_2), \quad q(x,0,z) = q(x,\omega_2,z).$$

Let u be a solution of problem $(6),(7)$. Set $v(y) = u_x(0,y) - P_2\, u(0,y)$. Then v is a solution of the problem
$$v' = q(0,y,\varphi(y)), \tag{8}$$
$$v(0) - v(\omega_2) = 0. \tag{9}$$

In other words the solvability of $(8),(9)$ is necessary for the solvability of problem $(6),(7)$. Problem $(8),(9)$ itself is ill-posed. It is solvable if and only if the following equality holds
$$\int\limits_0^{\omega_2} q\big(0,t,\varphi(t)\big)\,dt = 0.$$

Remark 3. The fulfillment of additional compatibility conditions is necessary for solvability of problem (1), (2), but by no means sufficient. Indeed, consider the two-dimensional problem

$$\begin{aligned} u_{1xy} &= u_2^3 - \cos x, \\ u_{2xy} &= -u_1^5 + \sin x, \end{aligned} \tag{10}$$

$$\begin{gathered} u_1(0,y) = 0, \quad u_1(\omega_1,y) = 0, \\ u_{1x}(x,0) = u_{1x}(x,\omega_2), \quad u_{2x}(x,0) = u_{2x}(x,\omega_2). \end{gathered} \tag{11}$$

Let us show that problem (10), (11) has at most one solution. Indeed, let

$$u(x,y) = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} \quad \text{and} \quad \widetilde{u}(x,y) = \begin{pmatrix} \widetilde{u}_1(x,y) \\ \widetilde{u}_2(x,y) \end{pmatrix}$$

be arbitrary solutions of problem (10), (11). Then, in view of (10), we have

$$\big(u_1(x,y) - \widetilde{u}_1(x,y)\big)_{xy} = u_2^3(x,y) - \widetilde{u}_2^3(x,y), \tag{12}$$

$$\big(u_2(x,y) - \widetilde{u}_2(x,y)\big)_{xy} = -\big(u_1^5(x,y) - \widetilde{u}_1^5(x,y)\big). \tag{13}$$

Multiply (12) by $u_2 - \widetilde{u}_2$, integrate over Ω. After integrating by parts and taking into account conditions (11), we arrive at the equality

$$-\int_0^{\omega_1}\int_0^{\omega_2} \big(u_1(x,y) - \widetilde{u}_1(x,y)\big)_x \big(u_2(x,y) - \widetilde{u}_2(x,y)\big)_y \, dy\, dx = \int_0^{\omega_1}\int_0^{\omega_2} \big(u_2^3(x,y) - \widetilde{u}_2^3(x,y)\big)\big(u_2(x,y) - \widetilde{u}_2(x,y)\big)\, dy\, dx. \tag{14}$$

Similarly, after multiplying (13) by $u_1 - \widetilde{u}_1$ and integrating over Ω, we get

$$-\int_0^{\omega_1}\int_0^{\omega_2} \big(u_2(x,y) - \widetilde{u}_2(x,y)\big)_y \big(u_1(x,y) - \widetilde{u}_1(x,y)\big)_x \, dy\, dx = -\int_0^{\omega_1}\int_0^{\omega_2} \big(u_1^5(x,y) - \widetilde{u}_1^5(x,y)\big)\big(u_1(x,y) - \widetilde{u}_1(x,y)\big)\, dy\, dx. \tag{15}$$

After subtracting (15) from (14) we arrive at the equality

$$\int_0^{\omega_1}\int_0^{\omega_2} \big(u_2^3(x,y) - \widetilde{u}_2^3(x,y)\big)\big(u_2(x,y) - \widetilde{u}_2(x,y)\big)\, dy\, dx + \int_0^{\omega_1}\int_0^{\omega_2} \big(u_1^5(x,y) - \widetilde{u}_1^5(x,y)\big)\big(u_1(x,y) - \widetilde{u}_1(x,y)\big)\, dy\, dx = 0.$$

The latter equality implies $u_k(x,y) \equiv \widetilde{u}_k(x,y)$ $(k = 1,2)$, i.e., $u = \widetilde{u}$. In other words, problem (10), (11) has at most one solution. Therefore, due to uniqueness, the only possible solution of problem (10), (11) should be independent of y. Consequently,

$$u(x) = \begin{pmatrix} \cos^{\frac{1}{2}} x \\ \sin^{\frac{1}{5}} x \end{pmatrix}$$

is the only possible solution of problem (10), (11). It is clear that u is a weak solution but not a classical one since u is not differentiable at points $x = \frac{\pi}{2}\, m$ $(m = 0, 1, 2, 3, 4)$. Thus problem (10), (11) has no (classical) solution despite the fact that the right-hand side of system (10) and the boundary values are analytic functions.

Consider the system

$$u_{xy} = f(x, y, u_x, u_y, u) + q(x, y, u). \tag{16}$$

Theorem 3. *Let f satisfy all of the conditions of Theorem 2, and $q(x, y, z)$ be an arbitrary continuous function such that*

$$\lim_{\|z\|\to+\infty} \frac{\|q(x, y, z)\|}{\|z\|} = 0 \tag{17}$$

uniformly on Ω. Then problem (16), (2) has at least one solution.

For the quasi-linear system

$$u_{xy} = P_1(x, y)u_x + P_2(x, y)u_y + P_0(x, y)u + q(x, y, u) \tag{18}$$

Theorem 2 immediately implies

Corollary 1. *Let problem $(5_0), (2_0)$ be well-posed, and let $q(x, y, z)$ be an arbitrary continuous function satisfying condition (17) uniformly on Ω. Then problem (18), (2) has at least one solution.*

Let $n = 2m$, $u = (v, w)$, and $v, w \in \mathbb{R}^m$. For the system

$$\begin{aligned} v_{xy} &= A_1(y)w_x + B_1(x)w_y + f_1(x, y, w) + q_1\big(x, y, v, w\big), \\ w_{xy} &= A_2(y)v_x + B_2(x)v_y + f_2(x, y, v) + q_2\big(x, y, v, w\big) \end{aligned} \tag{19}$$

consider the boundary conditions of Nicoletti type

$$w(0, y) = 0, \quad v(\omega_1, y) = 0, \quad w_x(x, 0) = 0, \quad v_x(x, \omega_2) = 0, \tag{20}$$

and the periodic boundary conditions

$$v(0, y) = v(\omega_1, y), \quad w(0, y) = w(\omega_1, y), \quad v_x(x, 0) = v_x(x, \omega_2), \quad w_x(x, 0) = w_x(x, \omega_2). \tag{21}$$

Here $f_i = \big(f_{ik}\big)_{k=1}^m \in C(\Omega \times \mathbb{R}^m; \mathbb{R}^m)$ $(i = 1, 2)$, $q_i \in C(\Omega \times \mathbb{R}^{2m}; \mathbb{R}^m)$ $(i = 1, 2)$, and $A_i \in C([0, \omega_2]; \mathbb{R}^{m\times m})$ and $B_i \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ are symmetric matrix functions.

Corollary 2. *Let $A_1 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $A_2 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $B_1 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ and $B_2 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ be positive semi-definite symmetric matrix functions, and let there exist $\delta > 0$ such that the following conditions hold:*

$$f_{1k}(x, y, w_1, \dots, w_m)\, w_k \geq \delta\, w_k^2 f_1(x, y, w) \cdot w \geq \delta\|w\|^2 \ \textit{ for } \ (x, y, w_1, \dots, w_m) \in \Omega \times \mathbb{R}^m, \tag{22}$$

$$f_{2k}(x, y, v_1, \dots, v_m)\, v_k \leq -\delta\, v_k^2 f_2(x, y, v) \cdot v \leq -\delta\|v\|^2 \ \textit{ for } \ (x, y, v_1, \dots, v_m) \in \Omega \times \mathbb{R}^m, \tag{23}$$

$$\lim_{\|v\|,\, \|w\|\to+\infty} \frac{\|q_1(x, y, v, w)\| + \|q_2(x, y, v, w)\|}{\|v\| + \|w\|} = 0 \quad \textit{uniformly on } \Omega. \tag{24}$$

Then problem (19), (20) has at least one solution.

Corollary 3. *Let $A_1 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $A_2 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $B_1 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ and $B_2 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ be positive definite symmetric matrix functions, and let there exist $\delta > 0$ such that conditions (22)–(24) hold. Then problem (19), (21) has at least one solution.*

Remark 4. In Theorem 2 it is assumed that the function $f(x,y,v,w,z)$ has at most linear growth with respect to the phase variables v, w and z. Corollaries 2 and 3 cover the case where the right-hand side of system (19) has an arbitrary growth order in some phase variables. As an example, consider the systems

$$\begin{aligned} v_{xy} &= y^2 w_x + (1+x^2) w_y + w + \sinh(w) + \sin(x^2\,y^3)\,w^{\frac{4}{5}}, \\ w_{xy} &= \sin^2 x\,v_y - 2v - \sinh(v^3) + \ln\big(1 + x^2y^2 + v^6 + w^8\big) \end{aligned} \tag{25}$$

and

$$\begin{aligned} v_{xy} &= (1+y^2) w_x + (1+x^4)\,x_y w + \sinh(w) + \sin(x^2\,y^3)\,w^{\frac{4}{5}}, \\ w_{xy} &= e^y\,v_x + (1+\sin^2 x)\,v_y - 2v - \sinh(v^3) + \ln\big(1 + x^2y^2 + v^6 + w^8\big). \end{aligned} \tag{26}$$

System (25) satisfies all of the conditions of Corollary 2, and system (26) satisfies all of the conditions of Corollary 3. Therefore, by Corollaries 2 and 3, problems (25), (20) and (26), (21) are solvable.

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