Nonlocal Boundary Value Problems for Second Order Linear Hyperbolic Systems

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In the rectangle $\Omega = [0,\omega_1] \times [0,\omega_2]$ consider the problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y), \tag{1}$$

$$\ell(u(\,\cdot\,,y)) = \varphi(y), \quad h(u_x(x,\,\cdot\,)) = \psi(x), \tag{2}$$

where $P_j \in C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q \in C(\Omega;\mathbb{R}^n)$, $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$, and $\ell : C([0,\omega_1];\mathbb{R}^n) \to \mathbb{R}^n$ and $h : C([0,\omega_2];\mathbb{R}^n) \to \mathbb{R}^n$ are bounded linear operators that are *commutative*, i.e., the operators ℓ and h satisfy the equality

$$\ell \circ h(z) = h \circ \ell(z) \quad \text{for} \quad z \in C(\Omega;\mathbb{R}^n). \tag{3}$$

One may think that the boundary conditions

$$\ell(u(\,\cdot\,,y)) = \varphi(y), \quad h(u(x,\,\cdot\,)) = \Psi(x) \tag{$\widetilde{2}$}$$

are more natural than conditions (2). All the more so, conditions $(\widetilde{2})$ obviously imply conditions (2). The main reason for studying problem $(1),(2)$ instead of problem $(1),(\widetilde{2})$ is that problem $(1),(\widetilde{2})$ is ill-posed, since functions φ and ψ should satisfy certain compatibility conditions. Indeed, if $u \in C(\Omega)$ is an arbitrary function satisfying conditions $(\widetilde{2})$, then, in view of (3), we have

$$\ell(\psi) = \ell \circ h(u) = h \circ \ell(u) = h(\varphi).$$

By a solution of problem $(1),(2)$ we understand a *classical* solution, i.e., a function $u \in C^{1,1}(\Omega)$ satisfying equation (1) and boundary conditions (2) everywhere in Ω.

Along with problem $(1),(2)$ consider its corresponding homogeneous problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y, \tag{1_0}$$

$$\ell(u(\,\cdot\,,y)) = 0, \quad h(u_x(x,\,\cdot\,)) = 0, \tag{2_0}$$

as well as the problems

$$v' = P_2(x,y^*)v, \tag{1_1}$$

$$\ell(v) = 0 \tag{2_1}$$

and

$$v' = P_2(x^*,y)v, \tag{1_2}$$

$$h(v) = 0. \tag{2_2}$$

Problems $(1_1),(2_1)$ are $(1_2),(2_2)$ called ***associated problems*** of problem $(1),(2)$. Notice that problem $(1_1),(2_1)$ (problem $(1_2),(2_2)$) is a boundary value problem for a linear ordinary differential equation depending on a parameter y^* (a parameter x^*).

The concept of $\boldsymbol{\sigma}$-associated problems for n-dimensional periodic problems was introduced in [4], and for two-dimensional Dirichlet type problems in [3].

Definition 1. Problem (1), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{1,1}(\Omega)} \leq M\Big(\|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])} + \|q\|_{C(\Omega)}\Big),$$

where M is a positive constant independent of φ, ψ and q.

Theorem 1. *Let problem* (1), (2) *be solvable for arbitrary* $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ *and* $\psi \in C([0,\omega_1];\mathbb{R}^n)$. *Then the problem*

$$z' = 0, \quad \ell(z) = 0 \tag{4}$$

has only the trivial solution.

Remark 1. If problem (4) has only the trivial solution, then problem $(1_0), (2_0)$ is equivalent to the homogeneous problem

$$u_{xy} = P_0(x,y)\,u + P_1(x,y)\,u_x + P_2(x,y)\,u_y, \tag{1_0}$$

$$\ell(u(\,\cdot\,,y)) = 0, \quad h(u(x,\,\cdot\,)) = 0. \tag{$\widetilde{2_0}$}$$

Theorem 2. *Let* P_j $(j = 1,2)$ *be constant matrices, let problem* $(1_1), (2_1)$ *have a nontrivial solution, and let the following conditions hold:*

$$h(P_0(x,\,\cdot\,)z(\,\cdot\,)) = h(P_0(x,\,\cdot\,))\,h(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_2];\mathbb{R}^n),$$
$$h(P_1 z(\,\cdot\,)) = P_1\,h(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_2];\mathbb{R}^n),$$
$$h(P_2 z(\,\cdot\,)) = P_2\,h(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_2];\mathbb{R}^n).$$

Then for solvability of problem (1), (2) *it is necessary that the problem*

$$v' = P_2 v + (P_0 + P_2 P_1)\Psi(x) + h(q(x,\,\cdot\,)),$$
$$\ell(v) = h(\varphi') - \ell(P_1\,\Psi)$$

is solvable, where Ψ *is a solution of the problem*

$$z' = \psi(x), \quad \ell(z) = h(\varphi).$$

Theorem 3. *Let* P_j $(j = 1,2)$ *be constant matrices, let problem* $(1_2), (2_2)$ *have a nontrivial solution, and let along with* (4) *the following conditions hold:*

$$\ell(P_0(\,\cdot\,,y)z(\,\cdot\,)) = \ell(P_0(\,\cdot\,,y))\,\ell(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_1];\mathbb{R}^n),$$
$$\ell(P_1 z(\,\cdot\,)) = P_1\,\ell(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_1];\mathbb{R}^n),$$
$$\ell(P_2 z(\,\cdot\,)) = P_2\,\ell(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_1];\mathbb{R}^n).$$

Then for solvability of problem (1), (2) *it is necessary that the problem*

$$v' = P_1 v + (P_0 + P_1 P_2)\varphi(y) + h(q(\,\cdot\,,y)), \tag{5}$$

$$h\big(v\big) = \ell(\psi) - h(P_2\,\varphi) \tag{6}$$

is solvable.

Remark 2. Solvability of the ill-posed nonhomogenous problem (5), (6) means additional compatibility conditions between the boundary values φ and ψ, matrices P_0, P_1 and P_2, and the vector function q. Indeed, consider the problem

$$u_{xy} = P_0(x, y)u + q(x, y), \tag{7}$$

$$u(0, y) = \varphi(y), \quad u_x(x, 0) = u_x(x, \omega_2). \tag{8}$$

Let u be a solution of problem (7), (8). Set $v(y) = u_x(0, y)$. Then v is a solution of the problem

$$v' = P_0(0, y)\,\varphi(y) + q(0, y), \tag{9}$$

$$v(0) = v(\omega_2). \tag{10}$$

In other words the solvability of (9), (10) is necessary for the solvability of problem (7), (8). Problem (9), (10) itself is ill-posed. It is solvable if and only if the following equality holds

$$\int_0^{\omega_2} \big(P_0(0, t)\,\varphi(t) + q(0, t)\big)\,dt = 0.$$

Remark 3. Solvability of the ill-posed nonhomogenous problem (5), (6) is necessary for solvability of problem (1), (2), but by no means sufficient. Indeed, consider the two-dimensional problem

$$\begin{aligned} v_{1xy} &= w - q_1(x), \\ w_{2xy} &= -v + q_2(x), \end{aligned} \tag{11}$$

$$\begin{gathered} v(0, y) = 0, \quad v(\omega_1, y) = 0, \\ v_x(x, 0) = v_x(x, \omega_2), \quad w_x(x, 0) = w_x(x, \omega_2). \end{gathered} \tag{12}$$

Let us show that the corresponding homogeneous problem has only the trivial solution. Let

$$\begin{pmatrix} v(x, y) \\ w(x, y) \end{pmatrix}$$

be an arbitrary solution of the homogeneous system

$$v_{1xy} = w, \tag{13}$$

$$w_{2xy} = -v, \tag{14}$$

satisfying conditions (12). Multiply (13) by w, integrate over Ω. After integrating by parts and taking into account conditions (12), we arrive at the equality

$$-\int_0^{\omega_1}\int_0^{\omega_2} v_x(x, y)w_y(x, y)\,dy\,dx = \int_0^{\omega_1}\int_0^{\omega_2} w^2(x, y)\,dy\,dx. \tag{15}$$

Similarly, after multiplying (14) by v and integrating over Ω, we get

$$-\int_0^{\omega_1}\int_0^{\omega_2} w_y(x, y)v_x(x, y)\,dy\,dx = -\int_0^{\omega_1}\int_0^{\omega_2} v^2(x, y)\,dy\,dx. \tag{16}$$

After subtracting (16) from (15) we arrive at the equality

$$\int_0^{\omega_1}\int_0^{\omega_2} (v^2(x, y) + w^2(x, y))\,dy\,dx = 0.$$

Consequently the homogeneous problem (13), (14), (12) has only the trivial solution. Therefore, problem (11), (12) has at most one solution. Hence, the only possible (ω_2*-periodic with respec to the second variable*) solution of problem (11), (12) should be independent of y. Consequently,

$$\begin{pmatrix} v(x,y) \\ w(x,y) \end{pmatrix} = \begin{pmatrix} q_1(x) \\ q_2(x) \end{pmatrix}$$

is the only possible solution of problem (11), (12). It is clear that u is a weak solution but not a classical one, if q_1 and q_2 are *nowhere differentiable* continuous functions.

Theorem 4. *Let the following conditions hold:*

(A_0) *problem* (3) *has only the trivial solution;*

(A_1) *problem* $(1_1), (2_1)$ *has only the trivial solution for every* $y^* \in [0, \omega_2]$*;*

(A_2) *problem* $(1_2), (2_2)$ *have only the trivial solution for every* $x^* \in [0, \omega_1]$.

Then problem (1), (2) *has the Fredholm property, i.e. the following assertions hold:*

(i) *problem* $(1_0), (2_0)$ *has a finite dimensional space of solutions;*

(ii) *if problem* $(1_0), (2_0)$ *has only the trivial solution, then problem* (1), (2) *is uniquely solvable, and its solution* u *admits estimate*

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])}\Big), \tag{17}$$

where M *is a positive constant independent of* φ, ψ *and* q.

Definition 2. Problem (1), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega;\mathbb{R}^n)$, and its solution u admits the estimate (17), where M is a positive constant independent of φ, ψ and q.

Theorem 5. *Let problem* (1), (2) *be well-posed. Then conditions* (A_1) *and* (A_2) *of Theorem 4 hold.*

Remark 4. Consider the problem

$$u_{xy} = p(x)\, u_x + p(x)\, u_y - p^2(x)\, u + q(x,y), \tag{18}$$

$$u(0,x) = 2u(\omega_1, y), \quad u_x(x,0) = u_x(x,0), \tag{19}$$

where $p \in C^\infty([0,\omega_1])$ is a *nonnegative* function and $q \in C^\infty(\Omega)$. Let

$$q(x,y) = p(x)\, \widetilde{q}(x,y).$$

Set: $I_p = \{x \in [0,\omega_1] :\ p(x) = 0\}$. Then:

(i) problem (18), (19) is well-posed if and only if $I_p = \varnothing$. Moreover, if $I_p = \varnothing$, then a unique solution of problem (18), (19) belongs to $C^\infty(\Omega)$;

(ii) if $\widetilde{q} \in L^\infty([0,\omega_1])$, then problem (18), (19) has a unique weak solution if and only if $\operatorname{mes} I_p = 0$, and has *infinite dimensional set* of *nonclassical* weak solutions otherwise. If $\widetilde{q} \in C([0,\omega_2])$ and $\operatorname{mes} I_p = 0$, then that unique weak solution is a classical solution;

(iii) If $\widetilde{q} \in C([0,\omega_2])$, then problem (18), (19) has a *unique classical* solution if and only if I_p is nowhere dense in $[0,\omega_1]$, and has *infinite dimensional set* of classical solutions otherwise;

(iv) problem (18), (19) has a *unique classical* solution and *infinite dimensional set* of *weak* solutions if I_p is a nowhere dense set of a positive measure;

(v) if $q(x,y)=1$ and $I_p\neq\varnothing$, then problem (18), (19) has *no classical solution* despite the fact that the coefficients of equation (18) belong to $C^\infty(\Omega)$.

Theorem 6. *Let conditions* (A_0), (A_1), (A_2) *of Theorem 4 hold, and let* $P_2\in C^{0,1}(\Omega)$ *be such that*

$$h(v)=0 \implies h(P_2(\,\cdot\,,y)v(\,\cdot\,))=0 \ \text{ for } \ y\in[0,\omega_2]$$

for every function $v\in C([0,\omega_2])$. *Then there exists* $\varepsilon>0$ *such that if*

$$\big\|P(x,y)+P_1(x,y)\,P_2(x,y)-P_{2\,y}(x,y)\big\|\leq\varepsilon \ \text{ for } \ (x,y)\in\Omega,$$

then problem (1), (2) *is well-posed. In particular, if*

$$P(x,y)+P_1(x,y)\,P_2(x,y)-P_{2\,y}(x,y)=\mathrm{O},$$

then the solution of problem (1), (2_0) *admits the representation*

$$u(x_1,x_2)=\int_0^{\omega_1}\int_0^{\omega_2}G_1(x,s,y)\,G_2(y,t,s)\,q(s,t)\,dt\,ds,$$

where G_j *is Green's matrix of problem* (1_j), (2_j) $(j=1,2)$.

Let $n=2m$, $u=(v,w)$, and $v,w\in\mathbb{R}^m$. For the system

$$\begin{aligned} v_{xy}&=A_1(y)w_x+B_1(x)w_y+Q_1(x,y)w+q_1(x,y),\\ w_{xy}&=A_2(y)v_x+B_2(x)v_y+Q_2(x,y)v+q_2(x,y,) \end{aligned} \tag{20}$$

consider the following boundary conditions of Nicoletti type

$$w(0,y)=0,\quad v(\omega_1,y)=0,\quad w_x(x,0)=0,\quad v_x(x,\omega_2)=0, \tag{21}$$

and the periodic boundary conditions

$$\begin{aligned} v(0,y)&=v(\omega_1,y), & w(0,y)&=w(\omega_1,y),\\ v_x(x,0)&=v_x(x,\omega_2), & w_x(x,0)&=w_x(x,\omega_2). \end{aligned} \tag{22}$$

Corollary 1. *Let* $A_1\in C([0,\omega_2];\mathbb{R}^{m\times m})$, $A_2\in C([0,\omega_2];\mathbb{R}^{m\times m})$, $B_1\in C([0,\omega_1];\mathbb{R}^{m\times m})$ *and* $B_2\in C([0,\omega_1];\mathbb{R}^{m\times m})$ *be positive semi-definite symmetric matrix functions, and let there exist* $\delta>0$ *such that the following conditions hold:*

$$Q_1(x,y)\,w\cdot w\geq\delta\|w\|^2 \ \text{ for } \ (x,y,w)\in\Omega\times\mathbb{R}^m, \tag{23}$$

$$Q_2(x,y)\,v\cdot v\leq-\delta\|v\|^2 \ \text{ for } \ (x,y,w)\in\Omega\times\mathbb{R}^m. \tag{24}$$

Then problem (20), (21) *is well-posed.*

Corollary 2. *Let* $A_1\in C([0,\omega_2];\mathbb{R}^{m\times m})$, $A_2\in C([0,\omega_2];\mathbb{R}^{m\times m})$, $B_1\in C([0,\omega_1];\mathbb{R}^{m\times m})$ *and* $B_2\in C([0,\omega_1];\mathbb{R}^{m\times m})$ *be positive definite symmetric matrix functions, and let there exist* $\delta>0$ *such that conditions* (23) *and* (24) *hold. Then problem* (20), (22) *is well-posed.*

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