Antiperiodic Problem for One Class of Nonlinear Partial Differential Equations

Sergo Kharibegashvili

Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University Tbilisi, Georgia

E-mail: kharibegashvili@yahoo.co

In the plane of variables x and t consider a nonlinear partial differential equation of the form

$$L_f u := \frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^4} + \lambda \frac{\partial^2 u}{\partial x^2} + f(u) = F, \tag{1}$$

where f, F are given, while u unknown function, $\lambda = const$.

For the equation (1) we consider the following antiperiodic problem: find in the domain D_T : $0 < x < l, 0 < t < T$ a solution $u = u(x,t)$ of the equation (1) according to the boundary conditions

$$u(x,0) = -u(x,T), \quad u_t(x,0) = -u_t(x,T), \ \ 0 \leq x \leq l, \tag{2}$$

$$\frac{\partial^i u}{\partial x^i}\,(0,t) = -\frac{\partial^i u}{\partial x^i}\,(l,t), \ \ 0 \leq t \leq T, \ \ i = 0,1,2,3. \tag{3}$$

Note that to the study of antiperiodic problems for nonlinear partial differential equations, having a structure different from (1), is devoted numerous literature (see, for example, [1–7] and the references therein).

Denote by $C^{1,2}(\overline{D}_T)$ the space of functions continuous in $\overline{D}_T$, having in $\overline{D}_T$ continuous partial derivatives $\frac{\partial^i u}{\partial t^i}$, $i = 1,2$, $\frac{\partial^j u}{\partial x^j}$, $j = 1,2,3,4$. Let

$$\begin{aligned} C^{1,2}_{-}(\overline{D}_T) := \Big\{ u \in C^{1,2}(\overline{D}_T) : \ \ & \frac{\partial^i u}{\partial t^i}\,(x,0) = -\frac{\partial^i u}{\partial t^i}\,(x,T), \ \ 0 \leq x \leq l, \ \ i = 0,1, \\ & \frac{\partial^j u}{\partial x^j}\,(0,t) = -\frac{\partial^j u}{\partial x^j}\,(l,t), \ \ 0 \leq t \leq T, \ \ j = 0,1,2,3 \Big\}. \end{aligned}$$

Consider the Hilbert space $W^{1,2}_{-}(D_T)$ as a completion of the classical space $C^{1,2}_{-}(\overline{D}_T)$ with respect to the norm

$$\|u\|^2_{W^{1,2}_{-}(D_T)} = \int\limits_{D_T} \Big[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 \Big]\, dx\, dt. \tag{4}$$

Remark 1. It follows from (4) that if $u \in W^{1,2}_{-}(D_T)$ then $u \in W^1_2(D_T)$ and $\frac{\partial^2 u}{\partial x^2} \in L_2(D_T)$. Here $W^1_2(D_T)$ is the well-known Sobolev space consisting of the elements $L_2(D_T)$, having up to the first order generalized derivatives from $L_2(D_T)$.

Below, for function f in the equation (1) we require that

$$f \in C(\mathbb{R}), \ \ |f(u)| \leq M_1 + M_2 |u|^{\alpha}, \ \ \alpha = const > 1, \ \ u \in \mathbb{R}, \tag{5}$$

where $M_i = const \geq 0$, $i = 1,2$.

Remark 2. As it is known, since the dimension of the domain $D_T \subset \mathbb{R}^2$ equals two, then the embedding operator
$$I : W_2^1(D_T) \to L_q(D_T)$$
is linear and compact operator for any fixed $q = const > 1$. At the same time the Nemitskii operator $K : L_q(D_T) \to L_2(D_T)$, acting by formula $Ku = f(u)$. where $u \in L_q(D_T)$, and function f satisfies the condition (5) is bounded and continuous, when $q \geq 2\alpha$. Therefore, if we take $q = 2\alpha$ then the operator
$$K_0 = KI : W_2^1(D_T) \to L_2(D_T)$$
will be continuous and compact. Whence, in particular, we have that if $u \in W_2^1(D_T)$, then $f(u) \in L_2(D_T)$ and from $u_n \to u$ in the space $W_2^1(D_T)$ it follows $f(u_n) \to f(u)$ in the space $L_2(D_T)$.

Remark 3. Let $u \in C_-^{1,2}(\overline{D}_T)$ be a classical solution of the problem (1)–(3). Multiplying the both sides of the equation (1) by an arbitrary function $\varphi \in C_-^{1,2}(\overline{D}_T)$ and integrating obtained equality over the domain D_T with taking into account that the functions from the space $C_-^{1,2}(\overline{D}_T)$ satisfy the boundary conditions (2) and (3), we get

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 \varphi}{\partial x^2} + \lambda\,\frac{\partial u}{\partial x}\,\frac{\partial \varphi}{\partial x}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \quad \forall\,\varphi \in C_-^{1,2}(\overline{D}_T). \tag{6}$$

We take the equality (6) as a basis of definition of a weak generalized solution of the problem (1)–(3).

Definition 1. Let a function f satisfy the condition (5). A function $u \in W_-^{1,2}(D_T)$ is named a weak generalized solution of the problem (1)–(3), if the integral equality (6) holds for any function $\varphi \in W_-^{1,2}(\overline{D}_T)$, i.e.

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 \varphi}{\partial x^2} + \lambda\,\frac{\partial u}{\partial x}\,\frac{\partial \varphi}{\partial x}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \quad \forall\,\varphi \in W_-^{1,2}(D_T). \tag{7}$$

Note that due to Remark 2 the integral $\int\limits_{D_T} f(u)\varphi\,dx\,dt$ in the left-hand side of the equality (7) is defined correctly since from $u \in W_-^{1,2}(D_T)$ it follows that $f(u) \in L_2(D_T)$, and since $\varphi \in L_2(D_T)$, then $f(u)\varphi \in L_1(D_T)$.

It is easy to see that if a weak generalized solution u of the problem (1)–(3) in the sense of Definition 1 belongs to the class $C_-^{1,2}(\overline{D}_T)$, then it is a classical solution to this problem.

Under fulfillment of the condition
$$\lambda \geq 0 \tag{8}$$
in the space $C_-^{1,2}(\overline{D}_T)$ together with the scalar product

$$(u,v)_0 = \int\limits_{D_T} \Big[uv + \frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 v}{\partial x^2}\Big]\,dx\,dt \tag{9}$$

with norm $\|\cdot\|_0 = \|\cdot\|_{W_-^{1,2}(D_T)}$, defined by the right-hand side of the equality (4), let us consider the following scalar product

$$(u,v)_1 = \int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 v}{\partial x^2} + \lambda\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\Big]\,dx\,dt \tag{10}$$

with the norm

$$\|u\|_1^2 = \int\limits_{D_T} \Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 + \lambda \Big(\frac{\partial u}{\partial x}\Big)^2 \Big]\, dx, dt, \tag{11}$$

where $u, v \in C_-^{1,2}(\overline{D}_T)$.

The following inequalities

$$c_1 \|u\|_0 \leq \|u\|_1 \leq c_2 \|u\|_0 \quad \forall\, u \in C_-^{1,2}(\overline{D}_T)$$

with positive constants c_1 and c_2, independent of u, are valid. Whence due to (8)–(11) it follows that if we complete the space $C_-^{1,2}(\overline{D}_T)$ with respect to the norm (11), then we obtain the same Hilbert space $W_-^{1,2}(D_T)$ with the equivalent scalar products (9) and (10). Using this circumstance, one can prove the unique solvability of the linear problem corresponding to (1)–(3), when $f = 0$, i.e. for any $F \in L_2(D_T)$ there exists a unique solution $u = L_0^{-1} F \in W_-^{1,2}(D_T)$ to this problem, where the linear operator

$$L_0^{-1} : L_2(D_T) \to W_-^{1,2}(D_T)$$

is continuous.

Remark 4. From the above reasoning, it follows that the nonlinear problem (1)–(3) is reduced equivalently to the functional equation

$$u = L_0^{-1}[f(u) - F] \tag{12}$$

in the Hilbert space $W_-^{1,2}(D_T)$.

Supposing that

$$\lim_{|u|\to\infty} \sup \frac{f(u)}{u} \leq 0, \tag{13}$$

it can be proved a priori estimate for the solution of the functional equation (12) in the space $W_-^{1,2}(D_T)$, whence, due to Remarks 2 and 4, it follows the existence of the solution of the equation (12), and, therefore, of the problem (1)–(3) in the specified space. Thus, the following theorem is valid.

Theorem 1. *Let the conditions* (5), (8) *and* (13) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1)–(3) *has at least one weak generalized solution* u *in the space* $W_-^{1,2}(D_T)$.

Note that the monotonicity of the function f can provide the uniqueness of the solution of the problem (1)–(3).

Theorem 2. *If the conditions* (5), (8) *are fulfilled and* f *is a non-strictly decreasing function, i.e.*

$$(f(s_2) - f(s_1))(s_2 - s_1) \leq 0 \quad \forall\, s_1, s_2 \in \mathbb{R}, \tag{14}$$

then for any $F \in L_2(D_T)$ *the problem* (1)–(3) *can not have more than one weak generalized solution in the space* $W_-^{1,2}(\overline{D}_T)$.

These theorems imply the following theorem.

Theorem 3. *Let the conditions* (5), (8) *and* (13), (14) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1)–(3) *has a unique weak generalized solution* u *in the space* $W_-^{1,2}(D_T)$.

Note that if the condition (13) is violated, then the problem (1)–(3) may be unsolvable. Indeed, there is valid the following theorem.

Theorem 4. *Let the function f satisfy the conditions* (5), (8) *and*

$$f(u) \leq -|u|^{\alpha} \quad \forall\, u \in \mathbb{R}, \ \ \alpha = const > 1, \tag{15}$$

and the function $F = \mu F_0$, where $F_0 \in L_2(D_T)$, $F_0 > 0$ in the domain D_T, $\mu = const > 0$. Then there exists a number $\mu_0 = \mu_0(F_0, \alpha) > 0$ such that for $\mu > \mu_0$ the problem (1)–(3) *can not have a weak generalized solution in the space $\overset{\circ}{W}{}^{1,2}_{-}(D_T)$.*

It is easy to see that when the condition (15) is fulfilled, then the condition (13) is violated.

Acknowledgements

The work was supported by the Shota Rustavely National Science Foundation, Grant # FR-21-7307.

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