Solutions of Some Type n-th Order Differential Equations

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1 Statement of the problem

We consider the following nth-order differential equation

$$\big(r(t)u^{(m)}\big)^{(n-m)}=\sum_{k=0}^{m}p_k u^{(k)},\ \ n\geq 2, \tag{1.1}$$

where $p_k\in C_{loc}([a;+\infty[)$ $(k=0,\dots,m)$,

$$\lim_{t\to+\infty}\frac{p_0(t)}{q(t)}=\sigma,\quad \sigma=\operatorname{sign}(p_0(a)), \tag{1.2}$$

$r(t)$ and $q(t)$ are positive twice differentiable on the set $[a;+\infty[$ functions, $C_{loc}([a;+\infty[)$ is the space of locally continuous functions on the interval $[a;+\infty[$, $L([a;+\infty[)$ is the Banach space of Lebesgue integrable functions.

In case $m=0$ we have the equation

$$\big(r(t)u^{(m)}\big)^{(n-m)}\pm qy=0,\ \ n\geq 2,$$

that was considered, and for which the asymptotic images of the solutions were obtained, in Hinton's work [2].

In case $s\equiv 1$ and $m=n-1$ the equation

$$u^{(n)}=\sum_{k=0}^{n-1}p_k(t)u^{(k)}$$

was considered in the work by I. T. Kiguradze [3], the corresponding asymptotic images of the solutions were obtained when various conditions were imposed on the coefficients.

The purpose of this work is to establish the asymptotic images of the solutions of equation (1.1) as $t\to+\infty$.

2 Main results

The following theorem has been obtained.

Theorem. *Let for equation* (1.1) *condition* (1.2) *and the following conditions be satisfied*

$$\Big(\frac{q}{r}\Big)^{\frac{1}{n}}\notin L([a;+\infty[), \tag{2.1}$$

$$\bigg(\frac{r'}{r}\cdot\Big(\frac{q}{r}\Big)^{-\frac{1}{n}}\bigg)'\in L([a;+\infty[),\quad \bigg(\frac{q'}{q}\cdot\Big(\frac{q}{r}\Big)^{-\frac{1}{n}}\bigg)'\in L([a;+\infty[), \tag{2.2}$$

$$\left(\frac{r'}{r}\right)^2 \cdot \left(\frac{q}{r}\right)^{-\frac{1}{n}} \in L([a;+\infty[), \quad \left(\frac{q'}{q}\right)^2 \cdot \left(\frac{q}{r}\right)^{-\frac{1}{n}} \in L([a;+\infty[), \tag{2.3}$$

$$\frac{p_{k-1}(t)}{q(t)} \cdot \left(\frac{q}{r}\right)^{\frac{k-1}{n}} \in L([a;+\infty[) \ \ (k=2,\dots,m), \quad \frac{p_m(t)}{r(t)q(t)} \cdot \left(\frac{q}{r}\right)^{\frac{m}{n}} \in L([a;+\infty[).$$

Then equation (1.1) *has a fundamental system of solutions* u_j $(j=1,\dots,n)$, *which admit the asymptotic images*

$$u_j^{k-1} = q(t)^{-\alpha_k} \cdot r(t)^{-\beta_k} \cdot \exp\left[\lambda_j \cdot \int_a^t \left(\frac{q}{r}\right)^{\frac{1}{n}}\right] \cdot \left[\lambda_j^{k-1} + o(1)\right] \ \ (k,j=1,\dots,n) \tag{2.4}$$

in which λ_j^0 *are the roots of the equation*

$$\lambda^n = \sigma. \tag{2.5}$$

To prove this theorem, the following transformations were applied to equation (1.1):

$$\begin{cases} u^{(i)}(t) = z_{i+1}(t), \ \ 0 \le i \le m-1, \\ u^{(m)}(t) = \dfrac{z_{m+1}(t)}{r(t)}, \\ (r(t)u^{(m)})^{(i-m)} = z_{i+1}(t), \ \ m+1 \le i \le n-1, \ \ m \ne n-1. \end{cases}$$

The following system of quasi-linear equations equivalent to equation (1.1) is obtained

$$\begin{cases} z'_{(i)}(t) = z_{i+1}(t), \ \ 1 \le i \le n-1, \ \ i \ne m, \\ z'_{(m)}(t) = \dfrac{z_{m+1}(t)}{r(t)}, \\ z'_n = p_0(t) z_1 + \displaystyle\sum_{i=1}^{m-1} p_i(t) \cdot z_{i+1} + \dfrac{p_m(t)}{r(t)} \cdot z_{m+1}. \end{cases} \tag{2.6}$$

Let's write system (2.6) in matrix form:

$$Z' = P \cdot Z, \tag{2.7}$$

where

$$P = (p_{ij})_1^n, \quad p_{ij} = \begin{cases} 1, & 1 \le i \le n-1, \ \ i \ne m, \ \ j = i+1, \\ \dfrac{1}{r(t)}, & i = m, \ \ j = i+1, \\ p_{i-1}, & i = n, \ \ 1 \le j \le m, \\ \dfrac{p_m}{r}, & i = n, j = m+1, \\ 0, & \text{otherwise}. \end{cases}$$

Further, the following transformation can be applied to system (2.7):

$$Z(t) = Q(t) \cdot W(t), \tag{2.8}$$

in which

$$Q(t) = \operatorname{diag}\left[q^{\alpha_1} r^{\beta_1} \cdots q^{\alpha_n} r^{\beta_n}\right].$$

As a result of transformation (2.8), we have a system

$$W' = \big[Q^{-1}PQ - Q^{-1}Q'\big] \cdot W. \tag{2.9}$$

Also we note that the following statements are true

$$Q^{-1} = \operatorname{diag}\Big[\frac{1}{q^{\alpha_1} r^{\beta_1}} \cdots \frac{1}{q^{\alpha_n} r^{\beta_n}}\Big],$$

$$P \cdot Q = (a_{ij})_1^n, \quad a_{ij} = \begin{cases} q^{\alpha_{i+1}} \cdot r^{\beta_{i+1}}, & 1 \le i \le n-1, \ \ i \ne m, \ \ j = i+1, \\ q^{\alpha_{m+1}} \cdot r^{\beta_{m+1}-1}, & i = m, \ \ j = i+1, \\ p_{i-1} \cdot q^{\alpha_i} \cdot r^{\beta_i}, & i = n, \ \ 1 \le j \le m, \\ p_m \cdot q^{\alpha_{m+1}} \cdot r^{\beta_{m+1}-1}, & i = n, \ \ j = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

$$Q^{-1}PQ = (b_{ij})_1^n, \quad b_{ij} = \begin{cases} q^{\alpha_{i+1}-\alpha_i} \cdot r^{\beta_{i+1}-\beta_i}, & 1 \le i \le n-1, \ \ i \ne m, \ \ j = i+1, \\ q^{\alpha_{m+1}-\alpha_m} \cdot r^{\beta_{m+1}-\beta_m-1}, & i = m, \ \ j = i+1, \\ p_{i-1} \cdot q^{\alpha_i-\alpha_n} \cdot r^{\beta_i-\beta_n}, & i = n, \ \ 1 \le j \le m, \\ p_m \cdot q^{\alpha_{m+1}-\alpha_n} \cdot r^{\beta_{m+1}-\beta_n-1}, & i = n, \ \ j = m+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$Q^{-1}Q = \frac{q'}{q} \cdot D_1 + \frac{r'}{r} \cdot D_2, \quad D_1 = \operatorname{diag}[\alpha_1, \dots, \alpha_n], \quad D_2 = \operatorname{diag}[\beta_1, \dots, \beta_n].$$

We choose α_i and β_i in such a way that

$$\alpha_2 - \alpha_1 = \alpha_3 - \alpha_2 = \cdots = \alpha_{m+1} - \alpha_m = \alpha_n - \alpha_{n-1} = 1 + \alpha_1 - \alpha_n = \tau_\alpha,$$
$$\beta_2 - \beta_1 = \beta_3 - \beta_2 = \cdots = \beta_{m+1} - \beta_m - 1 = \beta_n - \beta_{n-1} = \beta_1 - \beta_n = \tau_\beta.$$

It follows from the last equalities that $\tau_\alpha = \frac{1}{n}$, $\tau_\beta = -\frac{1}{n}$.
Then we have the following equalities:

$$\begin{cases} \alpha_1 - \alpha_n = \dfrac{1}{n} - 1\,, \\ \alpha_2 - \alpha_n = \dfrac{1}{n} - \dfrac{n-1}{n}\,, \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{m+1} - \alpha_n = \dfrac{1}{n} - \dfrac{n-m}{n}\,. \end{cases}$$

Then let's $Q^{-1}PQ = (\frac{q}{r})^{\frac{1}{n}} \cdot [K+V]$, where $K = (k_{ij})_1^n$, $V = (v_{ij})_1^n$,

$$k_{ij} = \begin{cases} 1, & 1 \le i \le n-1, \ \ j = i+1, \\ \sigma, & i = n, \ \ j = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad v_{ij} = \begin{cases} \dfrac{p_0}{q} - \sigma, & i = n, \ \ j = 1, \\ \dfrac{p_{i-1}}{q} \cdot \Big(\dfrac{q}{r}\Big)^{\frac{i-1}{n}}, & i = n, \ \ 2 \le j \le m, \\ \dfrac{p_m}{q} \cdot \Big(\dfrac{q}{r}\Big)^{\frac{m}{n}}, & i = n, \ \ j = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, system (2.9) turns into the following system

$$W' = \Big[\Big(\frac{q}{r}\Big)^{\frac{1}{n}} \cdot [K+V] - \frac{q'}{q} \cdot D_1 + \frac{r'}{r} \cdot D_2\Big] \cdot W. \tag{2.10}$$

We apply the following transformation to system (2.10) once again:

$$h(t)=\int\limits_a^t\Big(\frac{q(\varsigma)}{r(\varsigma)}\Big)^{\frac{1}{n}}\,d\varsigma. \tag{2.11}$$

Let, moreover, g be a function inverse of the function h and for all $t>a$, $g(h(t))=t$.

Since conditions (2.1)–(2.3) of the theorem are fulfilled, then $h(t)\to\infty$ as $t\to\infty$. We also have that $W(s)=Z(g(s))$. As a result of transformation (2.11), we obtain a system

$$W'=\big[K+V-\alpha(s)\cdot D_1+\beta(s)\cdot D_2\big]\cdot W,$$

in which

$$\alpha(s)=\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\,\frac{q'}{q}\,,\quad \beta(s)=\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\,\frac{r'}{r}.$$

It also follows from conditions (2.1)–(2.3) that

$$\int\limits_0^\infty|\alpha'(s)|\,ds=\int\limits_a^\infty\bigg|\Big(\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\frac{q'}{q}\Big)'\bigg|\,ds<+\infty$$

and

$$\int\limits_0^\infty\alpha^2(s)\,ds=\int\limits_a^\infty\Big(\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\Big(\frac{q'}{q}\Big)^2\Big)\,ds<+\infty.$$

Similar results are valid for $\beta(s)$.

From all that has been shown and taking into account the conditions of the theorem, we proved that the conditions of the well-known Levinson result are satisfied [1, Theorem 8.1], the equation (1.1) is in some sense asymptotically equivalent to the corresponding binomial differential equation of the nth order. Therefore, equation (1.1) has a fundamental system of solutions u_j ($j=1,\dots,n$), which admit the asymptotic images (2.4).

References

[1] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*. McGraw-Hill Book Co., Inc., New York–Toronto–London, 1955.

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[3] I. T. Kiguradze, *Some singular Boundary Value Problems for Ordinary Differential Equations.* (Russian) Izdat. Tbilis. Univ., Tbilisi, 1975.