Stability of Global Attractors for the Chafee–Infante Equation w.r.t. Boundary Disturbances

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We consider the following initial boundary-value problem:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)), \quad t>0, \quad x\in(0,l), \\ u\big|_{x=0} = d_1(t), \quad u\big|_{x=l} = d_2(t), \\ u\big|_{t=0} = u_0(x). \end{cases} \tag{1}$$

Here $u=u(t,x)$ is an unknown function, $f\in\mathbb{C}^1(\mathbb{R})$ is a given nonlinear function satisfying conditions

$$\begin{gathered} \exists\, C_1>0 \quad \forall\, s\in\mathbb{R} \quad |f(s)|\le C_1(1+|s|^3), \\ \exists\, C_2>0, \ \ \alpha>0 \quad \forall\, s\in\mathbb{R} \quad f(s)\cdot s \ge -\alpha s^4 - C_2, \\ \exists\, C_3>0 \quad \forall\, s\in\mathbb{R} \quad |f'(s)|\le C_3(1+|s|^2). \end{gathered} \tag{2}$$

We consider bounded $d=d_1,d_2$ as a boundary disturbances.

It is well-known [6] that the corresponding undisturbed problem ($d\equiv 0$)

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)), \\ u\big|_{x=0} = u\big|_{x=l} = 0, \\ u\big|_{t=0} = u_0(x), \end{cases} \tag{3}$$

for every $u_0\in X=L^2(0,l)$ has a unique weak solution defined on $[0,+\infty)$.

Such solutions generate semigroup $\{S(t):\ X\mapsto X\}_{t\ge 0}$ which has a global attractor $\Theta\subset X$ [6].

Definition. A compact set $\Theta\subset X$ is called a global attractor of a semigroup $\{S(t):X\mapsto X\}_{t\ge0}$ if

- $\forall\, t\ge 0 \quad \Theta=S(t)\Theta$ (invariance);
- $\forall\, r>0 \quad \sup\limits_{\|u_0\|\le r} \operatorname{dist}(S(t)u_0,\Theta)\to 0$ as $t\to\infty$ (attraction).

The structure of the global attractor of problem (2) can be rather complicated, but it is well understood and can be investigated by analytical and numerical methods [3, 5, 6].

In particular, the set Θ is bounded in $L^\infty(0,l)$ and in $H^2(0,l)$, and

$$\Theta = W^u(N),$$

where N is a set of stationary solutions of (3), and $W^u(N)$ is an unstable set emanating from N, i.e., Θ consist of points lying on complete trajectories $u(\,\cdot\,)$ of (3) such that

$$\operatorname{dist}(u(t),N)\to 0 \ \text{ as } \ t\to\infty.$$

Moreover, the global attractor is stable in the Lyapunov sense, i.e.,

$$\forall\,\varepsilon>0\quad\exists\,\delta>0\quad\forall\,\xi\quad\text{such that}\ \ \|\xi\|_\Theta:=\operatorname{dist}(\xi,\Theta)<\delta$$

we have that

$$\forall\,t\geq 0\quad\|S(t)\xi\|_\Theta<\varepsilon.$$

So, for the undisturbed problem (3), we have that all trajectories eventually get to any neighborhood of the stable invariant set Θ.

The natural question arises: does this limit behaviour remain true under the presence of disturbances? The problem is that the disturbed problem is non-autonomous, and we have no guarantee in general, that it's solutions converge to Θ as $t\to\infty$. But we can expect that such attractivity property are affected only slightly by disturbances of small magnitude [2]. In [1] it was given a positive answer for this question in the case of external disturbances, i.e. when bounded functions $d=d(t,x)$ appears in the right-hand part of equation (3).

This property, named robust stability with respect to (w.r.t.) disturbances, can be effectively described in the Input-to-State Stability (ISS) framework [4]. In this work we apply this approach to the case of boundary disturbances.

Let us introduce the following classes of functions:

$$\begin{aligned}
K&=\big\{\gamma:[0,+\infty)\mapsto[0,+\infty)\ |\ \ \gamma\text{ is continuous strictly increasing, }\gamma(0)=0\big\};\\
KL&=\big\{\beta:[0,+\infty)\times[0,+\infty)\mapsto[0,+\infty)\ |\ \ \beta\text{ is continuous}\big\},\\
&\qquad\forall\,t>0\quad\beta(\,\cdot\,,t)\in K,\quad\forall\,s>0\quad\beta(s,\,\cdot\,)\ \text{ is strictly decreasing to zero.}
\end{aligned}$$

We prove that for every $d=\{d_1,d_2\}\in L^\infty[0,+\infty)$ and for every $u_0\in X=L^2(0,l)$ problem (1) has a unique weak solution $u(t)=S_d(t,u_0)$ defined on $[0,+\infty)$.

We also prove that for a shift-invariant subset $U\subset L^\infty[0,+\infty)$ the family $\{S_d\}_{d\in U}$ generates the semiprocess family, i.e.,

$$S_d(t+h,u_0)=S_{d(\,\cdot\,+h)}(t,S_d(h,u_0)).$$

Our main results are the following:

Theorem 1. *The semiprocess family $\{S_d\}_{d\in U}$, generated by* (1), *is locally ISS w.r.t. Θ, i.e., there exists $r>0$, $\beta\in KL$, and $\gamma\in K$ such that for any $\|u_0\|_\Theta\leq r$ and $\|d\|_\infty\leq r$ it holds that*

$$\forall\,t\geq 0\quad\|S_d(t,u_0)\|_\Theta\leq\beta\big(\|u_0\|_\Theta,t\big)+\gamma(\|d\|_\infty).\tag{4}$$

Theorem 2. *The semiprocess family $\{S_d\}_{d\in U}$, generated by* (1), *satisfies the asymptotic gain* (*AG*) *property w.r.t. Θ, i.e. there exists $\gamma\in K$ such that $\forall\,u_0\in X$ $\forall\,d\in U$ it holds that*

$$\limsup_{t\to\infty}\|S_d(t,u_0)\|_\Theta\leq\gamma(\|d\|_\infty).\tag{5}$$

It should be noted that the methods of proving (4) and (5) are different. To prove (4), we use Lyapunov's function technique. To prove (5), we use results on upper semicontinuity of global attractors with respect to parameters.

References

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