On the Solvability of a Periodic Problem in an Infinite Stripe for Second Order Hyperbolic Equations

Otar Jokhadze, Sergo Kharibegashvili

Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia
E-mails: ojokhadze@yahoo.com; kharibegashvili@yahoo.co

In the infinite stripe $D_T := \{(x,t) \in \mathbb{R}^2,\ x \in \mathbb{R},\ 0 < t < T\}$ of the plane of independent variables x, t we consider the problem of finding a regular solution $u = u(x,t)$ of the hyperbolic equation

$$au_{tt} + 2bu_{tx} + cu_{xx} = f(x,t), \quad (x,t) \in D_T, \ \ a,b,c := const, \ \ a \neq 0, \tag{1}$$

satisfying the periodic boundary conditions with respect to the variable t

$$u(x,0) = u(x,T), \quad u_t(x,0) = u_t(x,T), \ \ x \in \mathbb{R} := (-\infty,+\infty). \tag{2}$$

For hyperbolic equations and systems time periodic problems have been the subject of research by many authors (see, for example, works [2–6] and the references therein), in which questions of existence, absence, uniqueness and representation of solutions are studied.

Assuming that

$$b^2 - ac > 0, \ \ f \in C^1(\overline{D}_T), \tag{3}$$

the regular solution $u \in C^2(\overline{D}_T)$ of the equation (1) can be represented in the form

$$u(x,t) = \frac{\lambda_2\varphi(x-\lambda_1 t) - \lambda_1\varphi(x-\lambda_2 t)}{\lambda_2-\lambda_1} + \frac{1}{\lambda_2-\lambda_1}\int\limits_{x-\lambda_2 t}^{x-\lambda_1 t}\psi(\tau)\,d\tau + \frac{1}{a(\lambda_2-\lambda_1)}\int\limits_{D_{x,t}} f(\xi,\tau)\,d\xi\,d\tau, \tag{4}$$

where λ_i, $i=1,2$ by virtue (3) are the different real roots of the quadratic equation $a\lambda^2-2b\lambda+c=0$ and $D_{x,t}$ is the triangular domain bounded by an axis Ox and characteristic lines of the equation (1) coming from the point $(x,t) \in D_T$ and

$$\varphi(x) := u(x,0), \quad \psi(x) := u_t(x,0), \ \ x \in \mathbb{R}.$$

By applying the representation (4), the problem (1), (2) is equivalently reduced to a system of functional equations

$$\begin{cases} \psi(x) + \dfrac{1}{\lambda_2-\lambda_1}\Big[\lambda_1\psi(x-\lambda_1T) - \lambda_2\psi(x-\lambda_2T) + \lambda_1\lambda_2\varphi'(x-\lambda_1T) - \lambda_1\lambda_2\varphi'(x-\lambda_2T)\Big] \\ \qquad = \dfrac{1}{a(\lambda_2-\lambda_1)}\displaystyle\int\limits_0^T \Big\{-\lambda_1 f\big[x-\lambda_1(T-\tau),\tau\big] + \lambda_2 f\big[x-\lambda_2(T-\tau),\tau\big]\Big\}\,d\tau, \\ \varphi'(x) + \dfrac{1}{\lambda_2-\lambda_1}\Big[-\lambda_2\varphi'(x-\lambda_1T) + \lambda_1\varphi'(x-\lambda_2T) - \psi(x-\lambda_1T) + \psi(x-\lambda_2T)\Big] \\ \qquad = \dfrac{1}{a(\lambda_2-\lambda_1)}\displaystyle\int\limits_0^T \Big\{f\big[x-\lambda_1(T-\tau),\tau\big] - f\big[x-\lambda_2(T-\tau),\tau\big]\Big\}\,d\tau. \end{cases} \tag{5}$$

In the notation $v := \big(\psi, \varphi'\big)$ we write the system of equations (5) in the form

$$v(x) + \sum_{i=1}^{2} A_i v(x - \lambda_i T) = F(x), \quad x \in \mathbb{R}, \tag{6}$$

where

$$A_1 := \frac{1}{\lambda_2 - \lambda_1} \begin{Vmatrix} \lambda_1 & \lambda_1\lambda_2 \\ -1 & -\lambda_2 \end{Vmatrix}, \quad A_2 := \frac{1}{\lambda_2 - \lambda_1} \begin{Vmatrix} -\lambda_2 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 \end{Vmatrix}, \tag{7}$$

and

$$F(x) := \frac{1}{a(\lambda_2 - \lambda_1)} \begin{Vmatrix} \int\limits_0^T \Big\{ -\lambda_1 f\big[x - \lambda_1(T-\tau), \tau\big] + \lambda_2 f\big[x - \lambda_2(T-\tau), \tau\big] \Big\}\, d\tau \\ \int\limits_0^T \Big\{ f\big[x - \lambda_1(T-\tau), \tau\big] - f\big[x - \lambda_2(T-\tau), \tau\big] \Big\}\, d\tau \end{Vmatrix}.$$

If we introduce the notations

$$\omega_i := A_i v, \quad i = 1, 2, \tag{8}$$

by virtue of (7) and taking into account the facts that: $A_1A_2 = A_2A_1 = O$ and $A_i^2 := -A_i$, $i = 1, 2$, from the equation (6) with respect to the unknown functions ω_i, $i = 1, 2$, we get the following independent from each other equations

$$\omega_i(x) - A_i\omega_i(x - \lambda_i T) = A_i F(x), \quad x \in \mathbb{R}, \quad i = 1, 2. \tag{9}$$

For arbitrary $\alpha, \beta \in \mathbb{R}$ let's introduce the following spaces:

$$C_{\alpha,\beta}(\mathbb{R}) := \Big\{ v \in C(\mathbb{R}) : \sup_{x \in (-\infty,0)} e^{-\alpha x}|v(x)| + \sup_{x \in (0,+\infty)} e^{-\beta x}|v(x)| < +\infty \Big\},$$

$$C^2_{\alpha,\beta}(\overline{D}_T) := \Big\{ u \in C^2(\overline{D}_T) : \sup_{(x,t) \in (-\infty,0)\times[0,T]} e^{-\alpha x}\big(|u(x,t)| + |u_t(x,t)|\big) + \sup_{(x,t) \in (0,+\infty)\times[0,T]} e^{-\beta x}\big(|u(x,t)| + |u_t(x,t)|\big) < +\infty \Big\},$$

$$C^1_{\alpha,\beta}(\overline{D}_T) := \Big\{ f \in C^1(\overline{D}_T), \sup_{x \in (-\infty,0)\times[0,T]} e^{-\alpha x}|f(x,t)| + \sup_{x \in (0,+\infty)\times[0,T]} e^{-\beta x}|f(x,t)| < +\infty \Big\},$$

and the notation

$$I_{\alpha,\beta} := \big[\min(\alpha,\beta), \max(\alpha,\beta)\big].$$

Remark 1. It is easy to check that from the equalities (8) the vector function v is uniquely determined if and only if

$$\lambda_1\lambda_2 = \frac{c}{a} \neq 0. \tag{10}$$

Throughout Theorems 1–4 formulated below we will assume that the condition (10) is satisfied. Based on Bochner's results [1] regarding to the functional equation (9) in the space $C_{\alpha,\beta}(\mathbb{R})$ there are proved the following:

Theorem 1. *If $\alpha\beta > 0$, then for any right-hand side $f \in C^1_{\alpha,\beta}(\overline{D}_T)$ the problem (1), (2) has a unique solution in the space $C^2_{\alpha,\beta}(\overline{D}_T)$.*

Theorem 2. *If $\alpha < 0$ and $\beta > 0$, then for any right-hand side $f \in C^1_{\alpha,\beta}(\overline{D}_T)$ there exists a solution of the problem (1), (2) in the space $C^2_{\alpha,\beta}(\overline{D}_T)$, besides the corresponding homogeneous problem has an infinite number of linearly independent solutions in the same space.*

Theorem 3. *If $\alpha > 0$ and $\beta < 0$, then the problem* (1), (2) *in the space $C^2_{\alpha,\beta}(\overline{D}_T)$ cannot have more than one solution and for its solvability it is necessary and sufficient that the function $f \in C^1_{\alpha,\beta}(\overline{D}_T)$ satisfy the following condition*

$$\Lambda_\gamma(f) = 0, \quad \gamma \in \mathbb{R},$$

where Λ_γ - is a well-defined linear functional on the space $C^1_{\alpha,\beta}(\overline{D}_T)$, depending on a real parameter γ.

Theorem 4. *If $\alpha\beta = 0$, then the problem* (1), (2) *is not solvable even in the Hausdorff's sense in the space $C^2_{\alpha,\beta}(\overline{D}_T)$, when $f \in C^1_{\alpha,\beta}(\overline{D}_T)$, i.e. the set of functions f from the space $C^1_{\alpha,\beta}(\overline{D}_T)$ for which the problem* (1), (2) *is solvable in the space $C^2_{\alpha,\beta}(\overline{D}_T)$ is not closed in the space $C^1_{\alpha,\beta}(\overline{D}_T)$.*

Remark 2. Note also that in the case, when the condition (10) is violated, i.e. when $c = 0$, to the necessary conditions for the solvability of the problem (1), (2) in space $C^2_{\alpha,\beta}(\overline{D}_T)$ there will be added the following condition

$$\int\limits_0^T f(x,\tau)\,d\tau = 0 \quad \forall\, x \in \mathbb{R},$$

imposed on the function $f \in C^1_{\alpha,\beta}(\overline{D}_T)$.

Acknowledgements

This work was supported by the Shota Rustaveli National Science Foundation, Grant # FR-21-7307.

References

[1] S. Bochner, Allgemeine lineare Differenzengleichungen mit asymptotisch konstanten Koeffizienten. (German) *Math. Z.* **33** (1931), no. 1, 426–450.

[2] S. S. Kharibegashvili and O. M. Dzhokhadze, Time-periodic problem for a weakly nonlinear telegraph equation with directional derivative in the boundary condition. (Russian) *Differ. Uravn.* **51** (2015), no. 10, 1376–1392; translation in *Differ. Equ.* **51** (2015), no. 10, 1369–1386.

[3] T. Kiguradze, On periodic in the plane solutions of nonlinear hyperbolic equations. *Nonlinear Anal.* **39** (2000), no. 2, Ser. A: Theory Methods, 173–185.

[4] T. Kiguradze, On bounded and time-periodic solutions of nonlinear wave equations. *J. Math. Anal. Appl.* **259** (2001), no. 1, 253–276.

[5] B. I. Ptashnik, *Ill-Posed Boundary Value Problems for Partial Differential Equations.* (Russian) Naukova Dumka, Kiev, 1984.

[6] O. Vejvoda, *Partial Differential Equations: Time-Periodic Solutions.* Martinus Nijhoff Publishers, Springer, Dordrecht, The Hague, 1982.