Nonoscillation Theory of Nonlinear Differential Equations of Emden–Fowler Type with Variable Exponents

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1 Introduction

Since the publication of the pioneering papers by Koplatadze [2, 3], Koplatadze and Kvinikadze [4] and Yoshida [5, 6] there has been an increasing interest in the qualitative study of nonlinear differential equations with variable exponents. See also Došlá and Fujimoto [1].

In this lecture we take up second order Emden–Fowler type differential equations of the form

$$\big(p(t)\varphi_{\alpha(t)}(x')\big)' + q(t)\varphi_{\beta(t)}(x) = 0, \tag{A}$$

for which it is assumed that

(a) the coefficients $p(t)$ and $q(t)$ are positive continuous functions on $I = [a, \infty)$, $a \geq 0$;

(b) the exponents $\alpha(t)$ and $\beta(t)$ are positive continuous functions on I which tend to the non-zero limits $\alpha(\infty)$ and $\beta(\infty)$, respectively, as $t \to \infty$ in the extended real number system;

(c) the symbol $\varphi_\gamma(t)$ with a positive continuous function $\gamma(t)$ on I denotes the operator on $C(I)$ defined by
$$\varphi_{\gamma(t)}(u(t)) = |u(t)|^{\gamma(t)} \operatorname{sgn} u(t), \ \ u \in C(I).$$

We are concerned exclusively with nonoscillatory solutions of equation (A), that is, those solutions $x(t)$ of (A) which are defined on an interval of the form $J = [T, \infty)$, $T \geq a$, and eventually positive or negative there. For any solution $x(t)$ of (A) we define

$$D_\alpha x(t) = p(t)\varphi_{\alpha(t)}(x'(t)),$$

and call it the *quasi-derivative* of $x(t)$. It is easy to see that if $x(t)$ is a nonoscillatory solution of (A) on J, then (A) implies that its quasi-derivative $D_\alpha x(t)$ is eventually monotone on J so that $x'(t)$ is eventually of constant sign, and this means that $x(t)$ is eventually monotone on J. Thus it turns out that both $D_\alpha x(t)$ and $x(t)$ have the limits as $t \to \infty$ in the extended real number system. The pair of these limits $(x(\infty), D_\alpha x(\infty))$ is referred to as the *terminal state* of the solution $x(t)$. The terminal state of $x(t)$ can be a crucial indicator of the asymptotic behavior of $x(t)$ as $t \to \infty$.

Given an equation of the form (A), consider the set of all terminal states of its nonoscillatory solutions. This set is divided into a finite number of subsets, each of which claims its own pattern (or type) of asymptotic behavior shared by all members of that subset. It is expected that all these patterns specific to (A), once precisely analyzed, will provide us with a deeper insight into the overall asymptotic behavior at infinity of solutions of (A). As the object of our investigation we choose two classes of equations of the form (A), equations of category I and category II, which are defined in terms of the integrals

$$I_p = \int_a^\infty p(t)^{-\frac{1}{\alpha(t)}}\, dt \quad \text{and} \quad I_q = \int_a^\infty q(t)\, dt,$$

as follows. Equation (A) is said to be of category I or of category II according to whether $I_p = \infty$ and $I_q < \infty$ or $I_p < \infty$ and $I_q = \infty$, respectively. In this work we focus our attention on equations of these two categories, leaving equations of the remaining categories for later studies. Equation (A) of category I is studied in Section 2, where it turns out that there are three different patterns of terminal states of solutions of (A). This means that the entirety of solutions of (A) can be divided into three groups whose members exhibit different astmptotic behaviors as $t \to \infty$. Our most important task is to answer the question about the existence of solutions of (A) having these three patterns of asymptotic behavior. As it turns out, the question is too difficult to gain a complete answer. Section 3 is devoted to the study of equation (A) of category II. This equation of new category can also be handled by way of the standard analysis as developed in Section 1. However, we present here a surprisingly convenient means named *Duality Principle* which makes it possible to derive the desired results for equations of category II *almost automatically* from the results already known for equations of category I.

2 Nonoscillatory solutions of equation (A) of category I

We start with equation (A) of category I. Use is made of the following functions:

$$P_\alpha(t) = \int_a^t p(s)^{-\frac{1}{\alpha(s)}}\, ds, \quad \rho(t) = \int_t^\infty q(s)\, ds, \quad t \geq a.$$

It is clear that $P_\alpha(t) \to \infty$ and $\rho(t) \to 0$ as $t \to \infty$.

Let $x(t)$ be any nonoscillatory solution of (A) on $J = [T, \infty)$, $T \geq a$. We may assume without loss of generality that $x(t) > 0$ on J. Then, it can be shown that $D_\alpha x(t) > 0$ on J, so that the terminal states of $x(t)$ is divided into the following three patterns:

I-(i) $\{x(\infty) = \infty,\ < D_\alpha x(\infty) < \infty\}$,

I-(ii) $\{x(\infty) = \infty,\ D_\alpha x(\infty) = 0\}$,

I-(iii) $\{0 < x(\infty) < \infty,\ D_\alpha x(\infty) = 0\}$.

A solution of (A) having the asymptotic pattern I-(i), I-(ii) or I-(iii) is named a *maximal solution*, an *intermediate solution* or a *minimal solution* of (A). Note that maximal and intermediate solutions are unbounded on J. It is important to recognize that the order of growth of a maximal solution $x(t)$ of (A) as $t \to \infty$ is precisely determined by the value $D_\alpha x(\infty)$ as follows:

$$D_\alpha x(\infty) = d \in (0, \infty) \implies \lim_{t\to\infty} \frac{x(t)}{P_\alpha(t)} = d^{\frac{1}{\alpha(\infty)}}.$$

On the other hand, as for an intermediate solution $x(t)$ of (A) nothing precise can be said about its growth at infinity except that it satisfies $\lim\limits_{t\to\infty} x(t)/P_\alpha(t) = 0$ because of $D_\alpha x(\infty) = 0$.

Our primary goal in this section is to characterize the existence of solutions with three different asymptotic patterns. More specifically, we want to find necessary and sufficient conditions for (A) to have maximal, intermediate and minimal solutions. This, however, is a difficult task in general. The first result concerns necessary conditions for the existence of maximal solutions of (A).

Theorem 2.1. *Let* (A) *be of category* I. *Suppose that* (A) *has a maximal solution* $x(t)$ *such that*

$$\lim_{t\to\infty} D_\alpha x(t) = d \ \text{ or equivalently } \ \lim_{t\to\infty} \frac{x(t)}{P_\alpha(t)} = d^{\frac{1}{\alpha(\infty)}} \ \text{ for some constant } \ d > 0. \tag{2.1}$$

(i) *If* $d > 1$, *then it holds that*

$$\int\limits_a^\infty q(t) P_\alpha(t)^{\beta(t)}\, dt < \infty. \tag{2.2}$$

(ii) *Let the condition* $\beta(\infty) < \infty$ *be added to (A). If* $0 < d \le 1$, *then* (2.2) *is satisfied.*

The second result gives sufficient conditions for (A) to have maximal solutions.

Theorem 2.2. *Let* (A) *be of category* I. *Suppose that* (2.2) *is satisfied.*

(i) *Equation* (A) *has a maximal solution* $x(t)$ *satisfying* (2.1) *for any given* $d < 1$.

(ii) *Equation* (A) *with* $\beta(\infty) < \infty$ *has a maximal solution* $x(t)$ *satisfying* (2.1) *for any given* $d \ge 1$.

From the above two theorems combined we have the following result characterizing the existence of maximal solutions for (A).

Theorem 2.3. *Let* (A) *be of category* I. *Assume that* $\beta(\infty) < \infty$. *Then,* (A) *has a maximal solution* $x(t)$ *satisfying* (2.1) *for any positive constant* d *if and only if* (2.2) *is satisfied.*

Let us turn our attention to minimal solutions $x(t)$ of equation (A) having the asymptotic pattern

$$\lim_{t\to\infty} D_\alpha x(t) = 0, \quad \lim_{t\to\infty} x(t) = \omega \ \text{ for some constants } \ \omega \neq 0. \tag{2.3}$$

Such solutions can also be handled in essentially the same way as maximal solutions, and we are led to the following results.

Theorem 2.4. *Let* (A) *be of category* I. *Suppose that* (A) *has a minimal solution* $x(t)$ *satisfying* (2.3) *for some non-zero constant* ω.

(i) *If* $|\omega| > 1$, *then it holds that*

$$\int\limits_a^\infty \big(p(t)^{-1}\rho(t)\big)^{\frac{1}{\alpha(t)}}\, dt < \infty. \tag{2.4}$$

(ii) *Let the condition* $\beta(\infty) < \infty$ *be added to* (A). *If* $0 < |\omega| \le 1$, *then* (2.4) *is satisfied.*

Theorem 2.5. *Let* (A) *be category* I. *Suppose that* (2.4) *is satisfied.*

(i) *Equation* (A) *has a minimal solution* $x(t)$ *satisfying* (2.3) *for any given* ω *with* $0 < |\omega| \le 1$.

(ii) *Equation* (A) *with* $\beta(\infty) < \infty$ *has a minimal solution* $x(t)$ *satisfying* (2.3) *for any given* ω *with* $|\omega| > 1$.

Theorem 2.6. *Let* (A) *be of category* I. *Assume that* $\beta(\infty) < \infty$. *Then,* (A) *has a minimal solution* $x(t)$ *satisfying* (2.3) *for any positive constant* ω *if and only if* (2.4) *is satisfied.*

The analysis of intermediate solutions seems to be extremely difficult. What we have been able to achieve so far is to prove the existence of such a solution $x(t)$ satisfying

$$\lim_{t\to\infty} x(t) = \infty \ \text{ and } \ \lim_{t\to\infty} \frac{x(t)}{P_\alpha(t)} = 0,$$

only for the sublinear case of equation (A). We call equation (A) *sublinear* if $\alpha(t)$ decreases to $\alpha(\infty) > 0$, $\beta(t)$ increases to $\beta(\infty) > 0$ and $\alpha(\infty) > \beta(\infty)$. Our result reads:

Theorem 2.7. *Let* (A) *be of category* I *and sublinear. There exists an intermediate solution of* (A) *if*

$$\int_a^\infty \left(p(t)^{-1}\rho(t)\right)^{\frac{1}{\alpha(t)}} dt = \infty \ \text{ and } \ \int_a^\infty q(t)P_\alpha(t)^{\beta(t)}\, dt < \infty.$$

3 Nonoscillatory solutions of equation (A) of category II

This section is concerned with equation (A) of category II. Naturally this equation of new category can also be analyzed by the method similar to that employed in Section 2 for equations of category I. Here we avoid the routine approach, but instead we introduce a surprisingly convenient means called *Duality Principle* that makes it possible to derive all the desired asymptotic results for category II equations almost automatically from the known results for category I equations.

Let there be given equation (A). Putting $y(t) = -p(t)\varphi_{\alpha(t)}(x'(t))$ split (A) into the cyclic differential system

$$x'(t) = -p(t)^{-\frac{1}{\alpha(t)}}\, \varphi_{\frac{1}{\alpha(t)}}(y(t)), \quad y'(t) = q(t)\varphi_{\beta(t)}(x(t)), \tag{3.1}$$

and eliminate $x(t)$ and $x'(t)$ from (3.1). We then obtain the following differential equation for $y(t)$:

$$\left(q(t)^{-\frac{1}{\beta(t)}}\varphi_{\frac{1}{\beta(t)}}(y')\right)' + p(t)^{-\frac{1}{\alpha(t)}}\varphi_{\frac{1}{\alpha(t)}}(y) = 0. \tag{B},$$

Equation (B) is called the *reciprocal equation* of (A). Equation (B) is of the same type as (A) with different exponents $\alpha^*(t) = 1/\beta(t)$ and $\beta^*(t) = 1/\alpha(t)$. It is clear that the assumption (b) required for equation (A) is also satisfied for equation (B). As is easily seen, (A) is the reciprocal equation of (B).

A simple but noteworthy relationship between (A) and (B) called the duality principle will play a vital role in the whole development of this section.

Duality Principle

If equation (A) is of category I (resp. category II), then equation (B) is of category II (resp. category I), and vice versa.

We start with equation (A) for $x(t)$ of category II. To gain information about the asymptotic properties of its solutions, proceed as follows. First, form the reciprocal equation (B) for $y(t)$. Since (B) is of category I, all the results obtained in Section 1 can be applied to (B) so that we have a list of the main theorems describing the asymptotic properties of solutions $y(t)$ of (B). All the theorems in the list need to be rewritten as the statements regarding solutions $x(t)$ of equation (A). The new theorems thus obtained provide the asymptotic results we want to establish for equations of category II.

In the process of rewriting it is imperative to make correct use of the precise relationship between the data on (A) and those on (B) which are generated by (3.1) or equivalently by $y(t) = -D_\alpha x(t)$ and $x(t) = D_{\frac{1}{\beta}} y(t)$. Some of the main results obtained for (A) of category II via the duality principle are mentioned below.

Classification of solutions

Let (A) be of category II. If $x(t)$ is a positive solution on J of (A), then $D_\alpha x(t) < 0$ on J and its terminal state is one of the three patterns:

II-(i): $0 < x(\infty) < \infty$, $D_\alpha x(\infty) = -\infty$;

II-(ii): $x(\infty) = 0$, $D_\alpha x(\infty) = -\infty$;

II-(iii): $x(\infty) = 0$, $-\infty < D_\alpha x(\infty) < 0$.

A solution satisfying II-(i), II-(ii) or II-(iii) are called, respectively, a maximal solution, an intermediate solution, or a minimal solution of equation (A) of category II.

Using the functions

$$\pi_\alpha(t) = \int_t^\infty p(s)^{-\frac{1}{\alpha(s)}}\, ds, \quad Q(t) = \int_a^t q(s)\, ds,$$

which satisfy $\pi_\alpha(t) \to 0$ and $Q(t) \to \infty$ as $t \to \infty$, it is easy to show that the asymptotic behavior of a maximal solution or a minimal solution can be expressed as

$$\lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} \frac{D_\alpha x(t)}{Q(t)} = -c^{\frac{1}{\alpha(\infty)}}, \ \text{ for some constant } \ c > 0, \tag{3.2}$$

or

$$\lim_{t\to\infty} D_\alpha x(t) = -d, \quad \lim_{t\to\infty} \frac{x(t)}{\pi_\alpha(t)} = d^{\frac{1}{\alpha(\infty)}} \ \text{ for some constant } \ d > 0. \tag{3.3}$$

Existence of maximal and minimal solutions

Only the rewritten versions of Theorems 2.3 and 2.6 applied to (B) are presented here. The condition $\beta(\infty) < \infty$ needed in Theorems 2.3 and 2.6 is dispensed with for an obvious reason.

Theorem 3.1. *Equation* (A) *of category* II *has a maximal solution satisfying* (3.2) *for any* $c > 0$ *if and only if*

$$\int_a^\infty \Big(p(t)^{-1} Q(t)\Big)^{\frac{1}{\alpha(t)}}\, dt < \infty.$$

Theorem 3.2. *Equation* (A) *of category* II *has a minimal solution satisfying* (3.3) *for any* $d > 0$ *if and only if*

$$\int_a^\infty q(t)\pi_\alpha(t)^{\beta(t)}\, dt < \infty.$$

Existence of an intermediate solution

Let (A) be of category II. If in addition (A) is sublinear, then so is its reciprocal equation (B) of category I. First, apply Theorem 2.5 on equation (A) of category I to (B) and formulate a proposition on the existence of intermediate solutions $y(t)$ of (B). Then, using Duality Principle, translate the result into a theorem on intermediate solutions $x(t)$ of (A). It should read as follows:

Theorem 3.3. *Sublinear equation* (A) *of category II has an intermediate solution if*

$$\int_a^\infty q(t)\pi_\alpha(t)^{\beta(t)}\,dt = \infty \ \ \textit{and} \ \ \int_a^\infty \Big(p(t)^{-1}Q(t)\Big)^{\frac{1}{\alpha(t)}}\,dt < \infty.$$

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