On Decomposition Method for Bitsadze–Samarskii Nonlocal Boundary Value Problem for Nonlinear Two-Dimensional Second Order Elliptic Equations

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The present note is devoted to the Bitsadze–Samarskii nonlocal boundary value problem for nonlinear two-dimensional second-order elliptic equations. The sequential and parallel domain decomposition algorithms are considered.

The different kinds of problems with nonlocal boundary conditions arise very often. Nonlocal boundary value problems are quite an interesting generalization of classical problems and at the same time, they are naturally obtained when constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology, etc. (see, for example, [1, 3, 5, 10] and the references therein).

The nonlocal problems for ordinary differential equations, elliptic and other models are studied in many works (see, for example, [1–3, 5–7, 18] and the references therein). One of the main publications in this direction is work [3] by A. Bitsadze and A. Samarskii, in which by means of the method of integral equations the theorems of existence and uniqueness of a solution for the second-order multi-dimensional elliptic equations are proved. There are given some classes of problems for which the proposed method also works.

Numerous scientific papers deal with the investigation and numerical solution of problems considered in [3] and their modifications and generalizations. Many scientific papers are devoted to the construction and investigation of discrete analogs of the above-mentioned models. One of the first among them was the work [6] where the iterative method of proving the existence of a solution for the Laplace equation was proposed. By the approach proposed in the work [6], the nonlocal problem reduced to the classical Dirichlet problems, which yields the possibility to apply the elaborated effective methods for the numerical resolution of these problems. After this work, many scientists have been investigating nonlocal problems by using the same or different methods for elliptic equations and, among them, nonlinear models as well (see, for example, [1,2,5,7–14,16] and the references therein). Nevertheless, there are still many open questions in this direction.

It is well known that, in order to find the approximate solutions, it is important to construct useful cost-effective algorithms. For constructing such algorithms, the method of domain decomposition has great importance (see, for example, [19] and the references therein). There are several reasons why the domain decomposition techniques might be attractive. Applying this method, the whole problem can be reduced to relative subproblems on the domains which are comparatively less in size than the one considered at the beginning. At the same time, it's worth noting that, in addition to the sequential count algorithm on each of these domains, it is often possible to apply a parallel count algorithm as well. In the works [10–14, 16] domain decomposition method based on the Schwarz alternative method [4] is given for the study of nonlocal problems for Laplace [11–14,16] and nonlinear elliptic equations [10].

It is well known how a great role takes place variational formulation of boundary problems in modern mathematics. This question for nonlocal elliptic problems is at the beginning of study so far (see, for example, [13, 14] and the references therein).

The results of this paper are partially published in the work [10].

The outline of this note is as follows. The Bitsadze–Samarskii nonlocal boundary value problem for the nonlinear-second order two-dimensional elliptic equation in a rectangle is considered. The convergence of the Schwarz-type iterative sequential algorithm as well as the same question for the parallel algorithm is studied.

In the plane Oxy, let us consider the rectangle $G = \{(x,y) | -a < x < 0,\ 0 < y < b\}$, where a and b are the given positive constants. We denote the boundary of the rectangle G by ∂G and the intersection of the line $x = t$ with the set $\overline{G} = G \cup \partial G$ by Γ_t correspondingly.

Consider the following nonlocal Bitsadze–Samarskii boundary value problem:

$$\begin{gathered} F\Big(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\Big) = 0, \\ u(x,y)\big|_{\Gamma} = 0, \\ u(x,y)\big|_{\Gamma_{-\xi}} = u(x,y)\big|_{\Gamma_0}, \end{gathered} \tag{1}$$

where $\Gamma = \partial G \setminus \Gamma_0$, $\xi \in (0,a)$; $u(x,y) \in C(\overline{G}) \cap C^2(G)$ is an unknown function, F is the analytic function of its arguments: u, $u_x = p$, $u_y = q$, $u_{xx} = r$, $u_{xy} = s$, $u_{yy} = t$ and

$$4F_r F_t - F_s^2 \geq const > 0, \quad F_u \leq 0.$$

For the problem (1) let's consider the following sequential iterative procedure:

$$\begin{gathered} F\Big(x, y, u_1^k, \frac{\partial u_1^k}{\partial x}, \frac{\partial u_1^k}{\partial y}, \frac{\partial^2 u_1^k}{\partial x^2}, \frac{\partial^2 u_1^k}{\partial x \partial y}, \frac{\partial^2 u_1^k}{\partial y^2}\Big) = 0, \quad (x,y) \in G_1, \\ u_1^k(x,y)\big|_{\Gamma^1} = 0, \quad u_1^k(x,y)\big|_{\Gamma_{-\xi_1}} = u_2^{k-1}(x,y)\big|_{\Gamma_{-\xi_1}}, \end{gathered} \tag{2}$$

$$\begin{gathered} F\Big(x, y, u_2^k, \frac{\partial u_2^k}{\partial x}, \frac{\partial u_2^k}{\partial y}, \frac{\partial^2 u_2^k}{\partial x^2}, \frac{\partial^2 u_2^k}{\partial x \partial y}, \frac{\partial^2 u_2^k}{\partial y^2}\Big) = 0, \quad (x,y) \in G_2, \\ u_2^k(x,y)\big|_{\Gamma^2} = 0, \quad u_2^k(x,y)\big|_{\Gamma_{-\xi}} = u_2^k(x,y)\big|_{\Gamma_0} = u_1^k(x,y)\big|_{\Gamma_{-\xi}}, \\ k = 1, 2, \dots . \end{gathered} \tag{3}$$

Here we utilize the following notations:

$$G_1 = \{(x,y) \mid \ -a < x < -\xi_1,\ \ 0 < y < b\}, \quad G_2 = \{(x,y)\ -\xi < x < 0,\ \ 0 < y < b\},$$

where $-\xi_1$ is a fixed point of the interval $(-\xi, 0)$, $\Gamma^1 = \partial G_1 \setminus \Gamma_{-\xi_1}$, $\Gamma^2 = \partial G_2 \setminus (\Gamma_{-\xi} \cup \Gamma_0)$ and $u_2^0(-\xi_1, y) \equiv 0$.

The iterative procedure (2), (3) reduces the nonlocal nonclassical problem (1) to the sequence of classical Dirichlet boundary value problems on every step of the iteration.

As we have already noted, algorithm (2), (3) for the solution of the problem (1) has a sequential form. Now, let us consider one more approach to the solution of the problem (1). In this case, the search for approximate solutions on domains G_1 and G_2 will be carried out not by means of a sequential algorithm, but in a parallel way.

Consider the following parallel iterative process:

$$F\Big(x,y,u_1^k,\frac{\partial u_1^k}{\partial x},\frac{\partial u_1^k}{\partial y},\frac{\partial^2 u_1^k}{\partial x^2},\frac{\partial^2 u_1^k}{\partial x\partial y},\frac{\partial^2 u_1^k}{\partial y^2}\Big)=0,\quad (x,y)\in G_1,$$
$$u_1^k(x,y)\big|_{\Gamma^1}=0,\quad u_1^k(x,y)\big|_{\Gamma_{-\xi_1}}=u_2^{k-1}(x,y)\big|_{\Gamma_{-\xi_1}},\tag{4}$$

$$F\Big(x,y,u_2^k,\frac{\partial u_2^k}{\partial x},\frac{\partial u_2^k}{\partial y},\frac{\partial^2 u_2^k}{\partial x^2},\frac{\partial^2 u_2^k}{\partial x\partial y},\frac{\partial^2 u_2^k}{\partial y^2}\Big)=0,\quad (x,y)\in G_2,$$
$$u_2^k(x,y)\big|_{\Gamma^2}=0,\quad u_2^k(x,y)\big|_{\Gamma_{-\xi}}=u_2^k(x,y)\big|_{\Gamma_0}=u_1^{k-1}(x,y)\big|_{\Gamma_{-\xi}},\tag{5}$$
$$k=1,2,\dots,$$

where $u_1^0(-\xi,0)\equiv u_2^0(-\xi_1,0)\equiv 0$.

The following statements are true.

Theorem 1. *The sequential iterative process* (2), (3) *converges to a solution of the problem* (1) *uniformly in the domain* $\overline{G}$.

Theorem 2. *The parallel iterative process* (4), (5) *converges to a solution of the problem* (1) *uniformly in the domain* $\overline{G}$.

Remark 1. In the case of the Poisson equation, in Theorem 1 the following estimations are valid too for the sequential iterative process (2), (3):

$$|u(x,y)-u_1^k(x,y)|\le Cq^{k-1},\quad (x,y)\in\overline{G}_1,$$
$$|u(x,y)-u_2^k(x,y)|\le Cq^{k-1},\quad (x,y)\in\overline{G}_2,$$

and

$$|u(x,y)-u_1^k(x,y)|\le Cq^{\frac{k}{2}-1},\quad (x,y)\in\overline{G}_1,$$
$$|u(x,y)-u_2^k(x,y)|\le Cq^{\frac{k}{2}-1},\quad (x,y)\in\overline{G}_2,$$

for the parallel iterative process (4), (5).

Here $q\in(0,1)$ and C are constants independent of functions: $u(x,y), u_1^k(x,y), u_2^k(x,y)$.

Remark 2. The Bitsadze–Samarskii nonlocal boundary value problem for the above-mentioned nonlinear equation by using iterative process analogical to (2), (3) at first was studied in [14].

Remark 3. Theorems analogous to the above Theorems 1 and 2 are valid for the sequential as well as parallel algorithms for multi-grid domain decomposition case too.

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