Lyapunov Stability of Time-Fractional Stochastic Volterra Equations

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Time-fractional stochastic differential models became popular in applications, and its analysis is presented in multiple highly cited monographs and articles, for example, [1–3, 5, 6].

The target of this report is a stochastic fractional-in-time Volterra equation defined with multiple deterministic and stochastic time scales:

$$dx(t) = \sum_{j=1}^{m} \Big[f_j(t, (H_{1j}x)(t))\,(dt)^{\alpha_j} + g_j(t, (H_{2j}x)(t))\,dB_j(t) \Big] \ \ (t \geq 0). \tag{1}$$

Here $f_j(\omega, t, v)$ and $g_j(\omega, t, v)$ are random functions, H_{1j} and H_{2j} are linear delay operators, $0 < \alpha_j \leq 1$, $dB_j(t)$ are Itô differentials generated by the standard scalar Wiener processes (Brownian motions) B_j, m is the number of the deterministic/stochastic time-scales and $x(t)$ is an unknown stochastic process on $\Re$ satisfying, in addition to (1), the initial condition

$$x(s) = \varphi(s) \ \ (s \leq 0), \tag{2}$$

where $\varphi(\omega, s)$ is some random function (not necessarily continuous). Throughout the paper we tacitly assume that

$$f_j(\,\cdot\,,\,\cdot\,, 0) = 0 \ \text{ and } \ g_j(\,\cdot\,,\,\cdot\,, 0) = 0 \ \ (P \otimes \mu)\text{-almost everywhere}$$

(μ is the Lebesgue measure on $\Re$), which simply means that $x \equiv 0$ satisfies Eq. (1) and the initial condition (2) with $\varphi \equiv 0$. A solution of the initial value problem (1), (2) is a progressively measurable stochastic process x almost surely satisfying (2) for μ-almost all $s \in \Re_-$ and the integral equation

$$x(t) - \varphi(0) = \sum_{j=1}^{m} \Big[\int_0^t \alpha_j (t-s)^{\alpha_j - 1} f_j(s, (H_{1j}x)(s))\,ds + \int_0^t g_j(s, (H_{2j}x)(s))\,dB_j(s) \Big]$$

for all $t \in \Re_+$. It is assumed that the initial value problem (1), (2) has a unique solution $x(t, \varphi)$ for all admissible φ (see Definition 1).

Below we keep fixed the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\Re}, P)$ satisfying the standard conditions [1] assuming, in addition, that $\mathcal{F}_t = \mathcal{F}_0$ for all $t \le 0$. All stochastic processes in this paper are supposed to be progressively measurable w.r.t. this stochastic basis or parts of it.

Basic notation used below:

- $\Re = (-\infty, \infty)$, $\Re_+ = [0, \infty)$, $\Re_- = (-\infty, 0)$.
- μ is the Lebesgue measure defined on $\Re$ or its subintervals.
- E is the expectation.
- $|\cdot|$ is the fixed norm in $\Re^n$ and $\|\cdot\|$ is the associated matrix norm $\|\cdot\|$.
- $B_j(t)$ $(t \in \Re_+,\ j = 1, \dots, m)$ are the standard scalar Brownian motions (Wiener processes).

The constants used below:

- $n \in N$ is the dimension of the phase space, i.e. the size of the solution vector.
- $m \in N$ is the number of the deterministic/stochastic time-scales.
- The indices i, j satisfy $1 \le i \le 2$, $1 \le j \le m$.
- $0 < \alpha_j \le 1$ define the time scales.
- p is a fixed real constant appearing in the p-stability we assume that $p \ge 2$ and $p > \alpha_j^{-1}$.

Let $J \subset \Re_+$. The following spaces of random variables and stochastic processes are used below as well:

- The space k_p^n consists of all n-dimensional, $\mathcal{F}_0$-measurable random variables $\{\xi :\ E|\xi|^p < \infty\}$.
- $\mathcal{L}_p(J, \Re^l)$ contains all progressively measurable l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that
$$\int_J E|x(t)|^p\,dt < \infty.$$
- For a given positive continuous function $\gamma(t)$, $t \in J$, the space $\mathcal{M}_p^\gamma(J, \Re^l)$ consists of all progressively measurable l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that
$$\sup_{t\in J} E|\gamma(t)x(t)|^p < \infty.$$
- For $l = n$ and $J = \Re_+$ we define $\mathcal{M}_p^\gamma \equiv \mathcal{M}_p^\gamma(\Re_+, \Re^n)$, and if, in addition, $\gamma = 1$, then we put $\mathcal{M}_p \equiv \mathcal{M}_p^1(\Re_+, \Re^n)$.
- The Banach space $\mathcal{U}$ is the direct product of $2m$ copies of the space $\mathcal{M}_p(\Re_+, \Re^l)$ equipped with the natural norms.

In the well-known definition of the stochastic Lyapunov stability below we assume that $\varphi \in \mathcal{M}_p(\Re_- \cup \{0\}, \Re^n)$.

Definition 1. Eq. (1) is called globally

- p-stable if there exists $c > 0$ such that
$$E|x(t,\varphi)|^p \le c \sup_{s\le 0} E|\varphi(s)|^p \ \text{ for all } \ t \in \Re_+;$$

- asymptotically p-stable if it is p-stable and, in addition,

$$\lim_{t\to\infty} E|x(t,\varphi)|^p = 0;$$

- exponentially p-stable if there exist $c > 0$ and $\beta > 0$ such that the inequality

$$E|x(t,\varphi)|^p \le c\exp\{-\beta t\} \sup_{s\le 0} E|\varphi(s)|^p \text{ for all } t \in \Re_+$$

holds.

To study Lyapunov stability of the solutions of Eq. (1), it is convenient to rewrite it as a multi-time scale stochastic Volterra equation with predefined controls:

$$dy(t) = \sum_{j=1}^{m} \Big[(F_j(y,u_{1j}))(t)\,(dt)^{\alpha_j} + (G_j(y,u_{2j}))(t)\,dB_j(t)\Big] \ (t \ge 0), \tag{3}$$

where $u_{ij} = u_{ij}(t,\omega)$ $(t \in \Re_+)$ belong to the space $\mathcal{M}_p(\Re_+, \Re^l)$, F_j and G_j are some nonlinear Volterra mappings. The way to construct u_{ij}, F_j and G_j is described in the paper [7]. Note that Eq. (3) only requires the initial condition for $t = 0$

$$y(0) = y_0 \in k_p^n. \tag{4}$$

Given $u_{ij} \in \mathcal{M}_p(\Re_+, \Re^l)$, by a solution of the control problem (3), (4) we understand a progressively measurable stochastic process $y(t)$ almost surely satisfying the initial condition (4) and the integral equation

$$y(t) - y_0 = \sum_{j=1}^{m} \Big[\int_0^t \alpha_j (t-s)^{\alpha_j - 1} F_j(y,u_{1j})(s)\,ds + \int_0^t G_j(y,u_{2j})(s)\,dB_j(s)\Big]$$

for all $t \in \Re_+$. Two integrals here are understood in the sense of Lebesgue and Itô, respectively. In the sequel, we will assume that the restrictions on the operators F_j and G_j ensure the existence of these integrals and existence and uniqueness of the solution $y(t, y_0, u)$ of the control problem (3), (4) for all $u_{ij} \in \mathcal{M}_p(\Re_+, \Re^l)$ and $y_0 \in k_p^n$.

The Lyapunov stability of the solutions of Eq. (1) will be, then, replaced by a particular version of the input-to-state stability, which is well-known in the control theory. Below, we call this version $\mathcal{M}_p^\gamma$-stability.

Definition 2. We say that Eq. (3) is $\mathcal{M}_p^\gamma$-stable if for all $y_0 \in k_p^n$ and $u_{ij} \in \mathcal{M}_p(\Re_+, \Re^l)$

- $y(\,\cdot\,, y_0, u) \in \mathcal{M}_p^\gamma$;

- there exists $K > 0$ such that

$$\|y(\,\cdot\,, y_0, u)\|_{\mathcal{M}_p^\gamma} \le K\big(\|y_0\|_{k_p^n} + \|u\|_{\mathcal{U}}\big).$$

Under some very natural conditions on γ (see [7] for the details) the $\mathcal{M}_p^\gamma$-stability of solutions of Eq. (3) implies p-stability, asymptotic p-stability and exponential p-stability of solutions of Eq. (1). This result is exploited in this report.

To study the property of $\mathcal{M}_p^\gamma$-stability for Eq. (3) it is convenient to start with choosing some simpler linear equation, which already has this property:

$$dy(t)=\sum_{j=1}^{m}\Big[\big((Q_jy)(t)+z_{1j}(t)\big)\,(dt)^{\alpha_j}+z_{2j}(t)\,dB_j(t)\Big]\ \ (t\in\Re_+). \tag{5}$$

Here $Q_j:\mathcal{M}_p\to\mathcal{L}_{p_j}(\Re_+,\Re^n)$ $(p_j>\frac{1}{\alpha_j})$ are k_p^1-linear operators, $z_{1j}\in\mathcal{L}_{p_j}(\Re_+,\Re^n)$ and $z_{2j}\in\mathcal{L}_2(\Re_+,\Re^n)$. Assuming the existence and uniqueness property for Eq. (5) for any initial condition (4) and using the linearity of Q_j, we obtain the following representation of its solutions:

$$y(t)=U(t)\chi(0)+(Wz)(t),$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, which is an $n\times n$-matrix whose columns satisfy this homogeneous equation and $U(0)=I_n$ and

$$W:\prod_{j=1}^{m}\big(\mathcal{L}_{p_j}(\Re_+,\Re^n)\times\mathcal{L}_2(\Re_+,\Re^n)\big)\to\mathcal{M}_p$$

is Green's operator for (5), $(Wz)(0)=0$ and Wz is a solution of Eq. (5) for any z from the domain of W. Using the solutions representation of the auxiliary equation we can regularize Eq. (3) by rewriting it as

$$y(t)=U(t)y_0+\sum_{j=1}^{m}\Big[\big(W_{1j}(-Q_jy+F_j(y,u_{1j}))\big)(t)+\sum_{j=1}^{m}(W_{2j}G_j(y,u_{2j}))(t)\Big]\ \ (t>0).$$

Given a continuous function $\gamma:\Re_+\to(0,\infty)$, an initial value $y_0=[y_{01},\to,y_{0n}]^T\in k_p^n$, a control $u=(u_{ij}:\, i=1,2,\ j=1,\dots,m)$, $u_{ij}\in\mathcal{M}_p(\Re_+,\Re^l)$, which produce the solution of Eq. (3)

$$y(t,y_0,u)=\big[y_1(t,y_0,u),\dots,y_n(t,y_0,u)\big]^T$$

and a nonnegative stopping time η, we define

- $\overline{y}_0=[\overline{y}_{01},\dots,\overline{y}_{0n}]^T$, where
$$\overline{y}_{0\nu}=(E|y_{0\nu}|^p)^{1/p}\equiv\|y_{0\nu}\|_{k_p^1};$$
- $\overline{y}^{\,\eta}=[\overline{y}_1^{\,\eta},\dots,\overline{y}_n^{\,\eta}]^T$, where
$$\overline{y}_\nu^{\,\eta}=\sup_{0\le t\le\eta}\big(E|\gamma(t)y_\nu(t,y_0,u)|^p\big)^{1/p},$$

so that $\overline{y}_\nu^{\,\eta}=\overline{y}_\nu^{\,\eta}(\gamma,p)$, $\overline{y}^{\,\eta}=\overline{y}^{\,\eta}(\gamma,p)$ and $\overline{y}_\nu^{\,\eta}=\overline{y}_\nu^{\,\eta}(\gamma,p)$ for $\nu=1,\dots,n$. These notations allow us to formulate and prove the main result of this report.

Theorem 1. *Suppose there exist a real $n\times n$-matrix C and two constants $K_1>0$ and $K_2>0$ such that I_n-C is inverse-positive and for any stopping time $0\le\eta<\infty$ the vector $\overline{y}^{\,\eta}=\overline{y}^{\,\eta}(\gamma,p)$ satisfies the matrix inequality*

$$\overline{y}^{\,\eta}\le C\overline{y}^{\,\eta}+K_1\overline{y}_0+K_2\|u\|_{\mathcal{U}}e_n\ \ (e_n=[1,\dots,1]^T\in\Re^n).$$

Then Eq. (3) is $\mathcal{M}_p^\gamma$-stable.

The proof of the theorem can be found in [7].

Using this theorem, one can conveniently study different kinds of Lyapunov stability of the solutions of Eq. (1), choosing an appropriate weight γ and an auxiliary equation (5).

The illustrative example below demonstrates applications of Theorem 1. The universal constant c_p used in the example comes from the following estimate:

$$E\bigg|\int\limits_0^t f(s)\,dB(s)\bigg|^{2p} \le c_p^{2p} E\bigg(\int\limits_0^t |f(s)|^2\,ds\bigg)^p \quad (t \in \Re_+,\ \ p \ge 1), \tag{6}$$

where $B(t)$ $(t \in \Re_+)$ is the standard scalar Brownian motion and $f(s)$ ia an arbitrary scalar, progressive measurable stochastic process on $\Re_+$; some explicit formulae for c_p can be found in the literature, for instance, in [4], where $c_p = 2\sqrt{12}p$, which, however, is not best possible, as evidently, $c_1 = 1$,

Example. Let $1 \le p < \infty$. Consider the following system of linear equations

$$dx(t) = -\sum_{j=1}^{m}\Big[A^{(j)}x(h_j(t))\,(dt)^{\alpha_j} + \sum_{\tau=1}^{m_j} A^{(j,\tau)}x(h_{j\tau}(t))\,d\mathcal{B}_j(t)\Big] \quad (t \ge 0), \tag{7}$$

where $A^{(j)} = (a_{sl}^{(j)})_{s,l=1}^n$, $j = 1,\dots,m$, $A^{(j,\tau)} = (a_{sl}^{(j,\tau)})_{s,l=1}^n$, $j = 1,\dots,m$, $\tau = 1,\dots,m_i$ are real $n \times n$-matrices and h_j, $h_{j\tau}$, $j = 1,\dots,m$, $\tau = 1,\dots,m_j$ are continuous functions such that $h_j(t) \le t$, $h_{j\tau} \le t$, $t \ge 0$, $j = 1,\dots,m$, $\tau = 1,\dots,m_j$, $0 < \alpha_j \le 1$, $j = 1,\dots,m$, $A^{(1)}$ is a diagonal matrix with the positive diagonal entries $a_\nu^{(1)}$ and $\alpha_1 = 1$.

Let C be the $n \times n$-matrix with the entries

$$c_{\nu\kappa} = \sum_{j=2}^{m}\bigg[|a_{\nu\kappa}^{(j)}|\bigg(\exp\{-\alpha_j\}\Big(\frac{\alpha_j}{a_{\nu\nu}^{(1)}}\Big)^{\alpha_j} + \Gamma\Big(\frac{\alpha_j+1}{a_{\nu\nu}^{(1)}}\Big)^{\alpha_j}\bigg)\bigg] + \sum_{j=1}^{m}\sum_{\tau=1}^{m_j} c_p\bigg[\frac{|a_{\nu\kappa}^{(j,\tau)}|}{\sqrt{2a_{\nu\nu}}}\bigg] \quad (\nu,\kappa = 1,\dots,n). \tag{8}$$

Then the system (7) will be globally $2p$-stable if the matrix $I_n - C$ defined by (8) is inverse-positive. Here c_p is the universal constant from the estimate (6).

In this case one uses the constant weigth function $\gamma(t) = 1$ and an ordinary scalar equation (5).

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