Positive Periodic Solutions for Functional Differential Equations with Super-Linear Growth

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Consider a functional differential equation

$$u'(t) = \ell(u)(t) + \lambda F(u)(t) \quad \text{for a. e. } t \in \mathbb{R}, \tag{1}$$

where $\ell : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a linear bounded operator, $F : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions, and $\lambda \in \mathbb{R}$ is a parameter. By an ω-periodic solution to the equation (1) we understand a locally absolutely continuous ω-periodic function $u : \mathbb{R} \to \mathbb{R}$ that satisfies the equation (1) almost everywhere in $\mathbb{R}$. We say that an ω-periodic solution u to (1) is positive if $u(t) > 0$ for $t \in \mathbb{R}$.

Notation 1.

$\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$C_\omega(\mathbb{R})$ is the Banach space of ω-periodic continuous functions $v : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|v\|_{C_\omega} = \max\big\{|v(t)| : \ t \in [0, \omega]\big\}.$$

$C_\omega(\mathbb{R}_+) = \{v \in C_\omega(\mathbb{R}) : \ v(t) \in \mathbb{R}_+ \text{ for } t \in \mathbb{R}\}$.

$L_\omega(\mathbb{R})$ is the Banach space of ω-periodic locally Lebesgue integrable functions $p : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|p\|_{L_\omega} = \int\limits_0^\omega |p(s)|\, ds.$$

$L_\omega(\mathbb{R}_+) = \big\{p \in L_\omega(\mathbb{R}) : \ p(t) \in \mathbb{R}_+ \text{ for a. e. } t \in \mathbb{R}\big\}$.

If $A : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a linear bounded operator, by $\|A\|$ we denote the norm of A.

Notation 2. Let $c \in (0, 1)$. Then $\mathcal{K}_c \subset C_\omega(\mathbb{R}_+)$ is a set of functions such that $cu(s) \le u(t)$ for $s, t \in \mathbb{R}$.

Definition 1. We say that an operator ℓ belongs to the set $\mathcal{U}_c^+$ if every function $u \in C_\omega(\mathbb{R})$ that is locally absolutely continuous and satisfies

$$u'(t) \ge \ell(u)(t) \quad \text{for a. e. } t \in \mathbb{R},$$

belongs to $\mathcal{K}_c$.

It can be easily seen that if $\ell \in \mathcal{U}_c^+$, then the only ω-periodic solution to the homogeneous equation

$$u'(t) = \ell(u)(t) \quad \text{for a. e. } t \in \mathbb{R} \tag{2}$$

is the trivial solution.

Now we formulate the assumptions laid on the nonlinear operator F.

(H.1) F transforms $C_\omega(\mathbb{R}_+)$ into $L_\omega(\mathbb{R}_+)$ and it is not the zero operator, i.e., there exists $x_0 \in C_\omega(\mathbb{R}_+)$ such that

$$\int_0^\omega F(x_0)(s)\,ds > 0.$$

(H.2) F is super-linear with respect to $\mathcal{K}_c$, i.e., there exists a Carathéodory function $\eta : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F(v)(t) \geq \eta(t, \|v\|_{C_\omega}) \quad \text{for a. e. } t \in \mathbb{R}, \ \ v \in \mathcal{K}_c$$

and

$$\lim_{x\to+\infty} \frac{1}{x} \int_0^\omega \eta(s,x)\,ds = +\infty.$$

(H.3) For every $0 \neq x \in \mathcal{K}_c$ there exists $\delta_x > 0$ such that for every $\delta \in (0, \delta_x]$ we have

$$\delta F(x)(t) \geq F(v)(t) \quad \text{for a. e. } t \in \mathbb{R} \ \text{ whenever } \ v \in \mathcal{K}_c, \quad v(t) \leq \delta x(t) \quad \text{for } t \in \mathbb{R}$$

and

$$\delta_0 \int_0^\omega F(x)(s)\,ds > \int_0^\omega F(\delta_0 x)(s)\,ds \quad \text{for some } \ \delta_0 \in (0, \delta_x).$$

Note that the assumptions (H.1) and (H.3) imply $F(0)(t) = 0$ for a. e. $t \in \mathbb{R}$.

Notation 3. Let $\lambda \in \mathbb{R}$. Then by $\mathcal{S}(\lambda)$ we denote the set of all positive ω-periodic solutions to (1) for corresponding λ.

Theorem 1. *Let $c \in (0,1)$ be such that $\ell \in \mathcal{U}_c^+$ and F satisfies* (H.1)–(H.3)*. Then there exists a critical value $\lambda_c \in (0, +\infty]$ such that*

(i) *Eq.* (1) *has at least one positive ω-periodic solution provided $\lambda \in (0, \lambda_c)$,*

(ii) *Eq.* (1) *has no positive ω-periodic solution provided $\lambda \notin (0, \lambda_c)$.*

Moreover,

$$\lim_{\lambda\to\lambda_c^-} \sup\big\{\|u\|_{C_\omega} : \ u \in \mathcal{S}(\lambda)\big\} = 0,$$
$$\lim_{\lambda\to 0^+} \inf\big\{\|u\|_{C_\omega} : u \in \mathcal{S}(\lambda)\big\} = +\infty.$$

Because the ω-periodic solutions to (1) belong to $\mathcal{K}_c$, the latter means that the solutions uniformly tends to $+\infty$ as λ tends to zero.

Suppose that the operator F includes a linear part, i.e.,

$$F(v)(t) = \widetilde{F}(v,v)(t) \quad \text{for a. e. } t \in \mathbb{R}, \ \ v \in C_\omega(\mathbb{R}), \tag{3}$$

where $\widetilde{F} : C_\omega(\mathbb{R})\times C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions and it is linear and nondecreasing in the first variable. Therefore, instead of (1) we consider the equation

$$u'(t) = \ell(u)(t) + \lambda\widetilde{F}(u, u)(t) \quad \text{for a. e. } t \in \mathbb{R},$$

where ℓ and λ are the same as in (1) and $\widetilde{F}$ is described above.

Theorem 2. *Let $c \in (0, 1)$ be such that $\ell \in \mathcal{U}_c^+$ and F given by* (3) *satisfies* (H.1)–(H.3). *Let, moreover, $\widetilde{F}(\cdot, 0) : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ be a non-zero operator. Then, $\lambda_c < +\infty$ and the equation*

$$u'(t) = \ell(u)(t) + \lambda_c\widetilde{F}(u, 0)(t) \quad \textit{for a. e. } t \in \mathbb{R} \tag{4}$$

has a positive solution u_c, the set of solutions to (4) *is one-dimensional (generated by u_c), and*

$$T_\lambda \in \mathcal{U}_c^+ \quad \textit{for } \lambda \in]0, \lambda_c[\,,$$

where

$$T_\lambda(v)(t) \overset{def}{=} \ell(v)(t) + \lambda\widetilde{F}(v, 0)(t) \quad \textit{for a. e. } t \in \mathbb{R}, \ \ v \in C_\omega(\mathbb{R}).$$

If $\widetilde{F}(\cdot, 0) : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a zero operator, then $\lambda_c = +\infty$.

Theorem 2 gives us a method how to calculate the precise value of λ_c in the cases where F includes a linear part. Indeed, define an operator $A : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ by

$$A(x)(t) \overset{def}{=} \int\limits_{t-\omega}^{t} G(t, s)\widetilde{F}(x, 0)(s)\, ds \quad \text{for } t \in \mathbb{R}, \ \ x \in C_\omega(\mathbb{R}),$$

where G is Green's function to the ω-periodic problem for (2). Then

$$u_c(t) = \lambda_c A(u_c)(t) \quad \text{for } t \in \mathbb{R},$$

i.e., $1/\lambda_c$ is the first eigenvalue to A corresponding to the positive eigenfunction u_c. Therefore, according to Krasnoselski's theory and Gelfand's formula,

$$\lambda_c = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{\|A^n\|}}\,.$$

Corollaries

Consider a differential equation with deviating arguments

$$u'(t) = -g(t)u(\sigma(t)) + \lambda p(t)\,\frac{u(\tau(t))[a_0 + a_1u(\mu_1(t)) + a_2u(\mu_2(t))u(\mu_3(t))]}{b_0 + b_1u(\nu(t))} \quad \text{for a. e. } t \in \mathbb{R}, \tag{5}$$

where

- $p, g \in L_\omega(\mathbb{R}_+)$, $p \not\equiv 0$, $g \not\equiv 0$,
- σ, τ, μ_i, ν are measurable ω-periodic functions,
- $a_i > 0$, $b_i > 0$ are constants.

Corollary 1. *Let*

$$\int\limits_{\widetilde{\sigma}(t)}^{t} g(s)\,ds \leq \frac{1}{e} \quad \textit{for a. e. } t \in [0,\omega], \tag{6}$$

where $\widetilde{\sigma}(t) = \sigma(t) - z\omega$ *if* $\sigma(t) \in [t+(z-1)\omega, t+z\omega)$ $(z \in \mathbb{Z})$, *and let*

$$\frac{a_1}{b_1}\exp\bigg(-e\int\limits_0^{\omega} g(s)\,ds\bigg) > \frac{a_0}{b_0}\,. \tag{7}$$

Then there exists a critical value $\lambda_c \in (0,+\infty)$ *such that* (5) *has a positive* ω*-periodic solution iff* $\lambda \in (0,\lambda_c)$. *Moreover,*

$$\lim_{\lambda\to\lambda_c^-} \sup\big\{\|u\|_{C_\omega} :\ u \in \mathcal{S}(\lambda)\big\} = 0, \quad \lim_{\lambda\to 0^+} \inf\big\{\|u\|_{C_\omega} :\ u \in \mathcal{S}(\lambda)\big\} = +\infty.$$

The condition (6) guarantees that the operator

$$\ell(v)(t) \overset{def}{=} -g(t)v(\sigma(t)) \quad \text{for a. e. } t \in \mathbb{R},\ \ v \in C_\omega(\mathbb{R})$$

belongs to the set $\mathcal{U}_c^+$ with

$$c = \exp\bigg(-e\int\limits_0^{\omega} g(s)\,ds\bigg),$$

and the condition (7) guarantees that the assumption (H.3) is fulfilled with F defined by

$$F(v)(t) \overset{def}{=} p(t)\,\frac{v(\tau(t))[a_0 + a_1 v(\mu_1(t)) + a_2 v(\mu_2(t))v(\mu_3(t))]}{b_0 + b_1 v(\nu(t))} \quad \text{for a. e. } t \in \mathbb{R},\ \ v \in C_\omega(\mathbb{R}).$$

Now consider a differential equation with deviating arguments

$$u'(t) = -g(t)u(\sigma(t)) + \lambda p(t)\,\frac{u^{1+n}(\tau(t))(a + u(\mu(t)))^m}{(b + u^k(\nu(t)))}\,, \tag{8}$$

where

- $p, g \in L_\omega(\mathbb{R}_+)$, $p \not\equiv 0$, $g \not\equiv 0$,
- σ, τ, μ, ν are measurable ω-periodic functions,
- $a > 0$, $b > 0$, $n > 0$, $m > 0$, $k > 0$.

Corollary 2. *Let* $n + m > k$, *and let*

$$\int\limits_{\widetilde{\sigma}(t)}^{t} g(s)\,ds \leq \frac{1}{e} \quad \textit{for a. e. } t \in [0,\omega],$$

where $\widetilde{\sigma}(t) = \sigma(t) - z\omega$ *if* $\sigma(t) \in [t+(z-1)\omega, t+z\omega)$ $(z \in \mathbb{Z})$. *Then* (8) *has a positive* ω*-periodic solution for every* $\lambda > 0$. *Moreover,*

$$\lim_{\lambda\to+\infty} \sup\big\{\|u\|_{C} :\ u \in \mathcal{S}(\lambda)\big\} = 0, \quad \lim_{\lambda\to 0^+} \inf\big\{\|u\|_{C} :\ u \in \mathcal{S}(\lambda)\big\} = +\infty.$$