On the Representation Formula of a Solution for a Class of Perturbed Controlled Neutral Functional Differential Equation

Nika Gorgodze

Department of Mathematics, Akaki Tsereteli State University
Kutaisi, Georgia
E-mail: nika.gorgodze@atsu.edu.ge

Ia Ramishvili

Department of Mathematics, Georgian Technical University
Tbilisi, Georgia
E-mail: ia.ramis@yahoo.com

Tea Shavadze

Ilia Vekua Institute of Applied Mathematics of Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia
E-mail: Tea.shavadze@gmail.com

The neutral functional-differential equation is a mathematical model of such a system whose behavior at a given moment depends on the velocity of the system in the past. In the paper an analytic relation between solutions of the original Cauchy problem and a corresponding perturbed problem is established for the controlled neutral functional-differential equation with the discontinuous initial condition, whose right-hand side is linear with respect to the prehistory of the phase velocity. In the representation formula of a solution the effects of perturbations of the delay parameter containing in the phase coordinates, of the initial and control functions are revealed. Such analytic relation plays an important role in proving the necessary conditions of optimality [1, 6]. Besides, such relation allows one to get an approximate solution of the perturbed equation and to carry out a sensitive analysis of mathematical models.

Let $I=[t_0,t_1]$ be a given interval. Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$ and let $O\subset\mathbb{R}^n$, $U\subset\mathbb{R}^r$ be convex open sets; let $\sigma>0$ and $\tau_2>\tau_1>0$ be given numbers, with $t_0+\max\{\sigma,\tau_2\}<t_1$. Suppose that the $n\times n$ -dimensional matrix function $A(t,x,y)$ is continuous on the set $I\times O^2$ and continuously differentiable with respect to x^i, $i=1,2,\dots,n$ and y^j, $j=1,2,\dots,n$; moreover, there exists $M_1>0$ such that

$$|A(t,x,y)|+\sum_{i=1}^{n}|A_{x^i}(\,\cdot\,)|+\sum_{j=1}^{n}|A_{y^j}(\,\cdot\,)|\le M_1\quad\forall\,(t,x,y)\in I\times O\times O.$$

Let the n-dimensional function $f(t,x,y,u)$ be continuous on the set $I\times O^2\times U$ and continuously differentiable with respect to x, y, u; moreover, there exists $M_2>0$ such that

$$|f(t,x,y,u)|+|f_x(\,\cdot\,)|+|f_y(\,\cdot\,)|+|f_u(\,\cdot\,)|\le M_2\quad\forall\,(t,x,y,u)\in I\times O^2\times U.$$

Further, denote by Φ and Ω the sets of continuous differentiable functions $\varphi(t)\in O$, $t\in[\widehat{\tau},t_0]$, where $\widehat{\tau}=t_0-\max\{\sigma,\tau_2\}$ and measurable functions $u(t)\in U$, $t\in I$, respectively, with the set $\operatorname{cl}u(I)$ is compact and $\operatorname{cl}u(I)\subset U$.

To each element
$$\mu=(\tau,x_0,\varphi(t),u(t))\in\Lambda=(\tau_1,\tau_2)\times O\times\Phi\times\Omega$$
we assign the quasi-linear neutral functional-differential equation
$$\dot x(t)=A(t,x(t),x(t-\tau))\dot x(t-\sigma)+f\big(t,x(t),x(t-\tau),u(t)\big),\ \ t\in I \tag{1}$$
with the initial condition
$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0),\quad x(t_0)=x_0. \tag{2}$$
Condition (2) is called the discontinuous initial condition because in general $x(t_0)\neq\varphi(t_0)$. Discontinuity at the initial moment t_0 may be related to the instant change in a dynamical process (for example, change of an investment, environment and so on).

Definition. Let $\mu\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in I_1=[\widehat\tau,t_1]$ is called a solution of equation (1) with condition (2) or a solution corresponding to the element μ and defined on the interval I_1 if it satisfies condition (2) and is absolutely continuous on the interval I and satisfies equation (1) almost everywhere on I.

Let us introduce the notations:
$$|\mu|=|\tau|+|x_0|+\|\varphi\|_1+\|u\|,\quad \Lambda_\varepsilon(\mu_0)=\big\{\mu\in\Lambda:\ |\mu-\mu_0|\le\varepsilon\big\},$$
where
$$\|\varphi\|_1=\sup\big\{|\varphi(t)|+|\dot\varphi(t)|:\ t\in I_1\big\},\quad \|u\|=\sup\big\{|u(t)|:\ t\in I\big\},$$
$\varepsilon>0$ is a fixed number and $\mu_0=(\tau_0,x_{00},\varphi_0(t),u_0(t))\in\Lambda$ is a fixed element; furthermore,
$$\delta\tau=\tau-\tau_0,\quad \delta x_0=x_0-x_{00},\quad \delta\varphi(t)=\varphi(t)-\varphi_0(t),\quad \delta u(t)=u(t)-u_0(t),$$
$$\delta\mu=\mu-\mu_0=(\delta\tau,\delta x_0,\delta\varphi(t),\delta u(t)).$$

Let $x(t;\mu_0)$ be a solution corresponding to the element $\mu_0\in\Lambda$ and defined on the interval I_1. Then there exists a number $\varepsilon_1>0$ such that to each element $\mu=\mu_0+\delta\mu\in\Lambda_{\varepsilon_1}(\mu_0)$ corresponds a solution $x(t;\mu)$, $t\in I_1$, [2, 6], i.e. Cauchy's perturbed problem has a solution, defined on the interval I_1.

Theorem 1. *Let $x_0(t):=x(t;\mu_0)$ be a solution corresponding to the element $\mu_0=(\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ and defined on the interval I_1, with $t_0+\tau_0\notin\{t_1-\sigma,t_1-2\sigma,\dots\}$. Moreover, let the function $u_0(t)$ be continuous at the point $t_0+\tau_0$. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta>0$ such that for arbitrary $\mu\in\Lambda_{\varepsilon_2}(\mu_0)$ on the interval $[t_1-\delta,t_1]\subset[t_0+\tau_0,t_1]$ the following representations hold:*
$$x(t;\mu)=x_0(t)+\delta x(t;\delta\mu)+o(t;\delta\mu), \tag{3}$$
$$\delta x(t;\delta\mu)=\Psi(t_0;t)\delta x_0+\int\limits_{t_0-\sigma}^{t_0}Y(\xi+\sigma;t)A[\xi+\sigma]\dot\delta\varphi(\xi)\,d\xi$$
$$+\int\limits_{t_0-\tau_0}^{t_0}Y(\xi+\tau_0;t)\Big\{\frac{\partial}{\partial y}\big[A[\xi+\tau_0]\dot x_0(\xi+\tau_0-\sigma)\big]+f_y[\xi+\tau_0]\Big\}\delta\varphi(\xi)\,d\xi+\int\limits_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$
$$-\Big\{Y(t_0+\tau_0;t)\big(\widehat A\dot x_0(t_0+\tau_0-\sigma)+\widehat f\big)+\int\limits_{t_0}^{t}Y(\xi;t)\Big(\frac{\partial}{\partial y}\big[A[\xi]\dot x_0(\xi-\sigma)\big]+f_y[\xi]\Big)\dot x_0(\xi-\tau_0)\,d\xi\Big\}\delta\tau. \tag{4}$$

Here,

$$A[\xi]=A(\xi,x_0(\xi),x_0(\xi-\tau_0)),\quad f_y[\xi]=f_y\big(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi)\big),$$
$$\frac{\partial}{\partial y}\big[A[\xi]\dot x_0(\xi-\sigma)\big]=\frac{\partial}{\partial y}\big[A(t,x,y)\dot y_0(\xi-\sigma)\big]_{x=x_0(\xi),\,y=x_0(\xi-\tau_0)},$$
$$\widehat A=A\big(t_0+\tau_0,x_0(t_0+\tau_0),x_{00}\big)-A\big(t_0+\tau_0,x_0(t_0+\tau_0),\varphi_0(t_0)\big),$$
$$\widehat f=f\big(t_0+\tau_0,x_0(t_0+\tau_0),x_{00},u_0(t_0+\tau_0)\big)-f\big(t_0+\tau_0,x_0(t_0+\tau_0),\varphi_0(t_0),u_0(t_0+\tau_0)\big);$$

$\Psi(\xi;t)$ and $Y(\xi;t)$ are $n\times n$ matrix functions satisfying the system

$$\begin{cases}\Psi_\xi(\xi;t)=-Y(\xi;t)\Big\{\dfrac{\partial}{\partial x}\big[A[\xi]\dot x_0(\xi-\sigma)\big]+f_x[\xi]\Big\}\\ \qquad -Y(\xi+\tau_0;t)\Big(\dfrac{\partial}{\partial y}\big[A[\xi+\tau_0]\dot x_0(\xi+\tau_0-\sigma)\big]+f_y[\xi+\tau_0]\Big),\\ Y(\xi;t)=\Psi(\xi;t)+Y(\xi+\sigma;t)A[\xi+\sigma],\\ \qquad \xi\in(t_0,t),\ \ t\in(t_0,t_1]\end{cases}\tag{5}$$

and the condition

$$\Psi(\xi;t)=Y(\xi;t)=\begin{cases}E, & \xi=t,\\ \Theta, & \xi>t,\end{cases}\tag{6}$$

where E is the identity matrix and Θ is the zero matrix.

Some Comments

The function $\delta x(t;\delta\mu)$ in (3) is called the first variation of the solution $x_0(t)$. The expression (4) is called the local variation formula of the solution. The term "variation formula of the solution" has been introduced by R. V. Gamkrelidze and proved for ordinary differential equation in [6].

The addend $\Psi(t_0;t)\delta x_0$ in formula (4) is the effect of perturbation of the initial vector x_{00}.

The expression

$$\int\limits_{t_0-\sigma}^{t_0}Y(\xi+\sigma;t)A[\xi+\sigma]\dot\delta\varphi(\xi)\,d\xi$$
$$+\int\limits_{t_0-\tau_0}^{t_0}Y(\xi+\tau_0;t)\Big\{\frac{\partial}{\partial y}\big[A[\xi+\tau_0]\dot x_0(\xi+\tau_0-\sigma)\big]+f_y[\xi+\tau_0]\Big\}\delta\varphi(\xi)\,d\xi$$

in formula (4) is the effect of perturbation of the initial function $\varphi_0(t)$.

The addend

$$\int\limits_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$

in formula (4) is the effect of perturbation of the control function $u_0(t)$.

The expression

$$\Big\{Y(t_0+\tau_0;t)\big(\widehat A\dot x_0(t_0+\tau_0-\sigma)+\widehat f\big)+\int\limits_{t_0}^{t}Y(\xi;t)\Big(\frac{\partial}{\partial y}\big[A[\xi]\dot x_0(\xi-\sigma)\big]+f_y[\xi]\Big)\dot x_0(\xi-\tau_0)\,d\xi\Big\}\delta\tau$$

in formula (4) is the effect of perturbation of the delay τ_0, where $Y(t_0+\tau_0;t)(\widehat{A}\dot{x}_0(t_0+\tau_0-\sigma)+\widehat{f})$ is the effect of the discontinuous initial condition (2). If $x_0(t_0)=\varphi_0(t_0)$, then $\widehat{A}=0$ and $\widehat{f}=0$.

Formula (3) allows us to obtain an approximate solution of the perturbed equation in the analytical form on the interval $[t_1-\delta,t_1]$. In fact, for a small $|\delta\mu|$ from (3) it follows

$$x(t;\mu)\approx x_0(t)+\delta x(t;\delta\mu),$$

where $\delta x(t;\delta\mu)$ has the form (4). We note that to construct $\delta x(t;\delta\mu)$ it is sufficient to find a solution to the linear problem (5), (6).

Theorem 1 is proved by the scheme given in [2,6]. The case when $A(t,x,y)=A(t)$ is considered in [2,3,6] and the case when $A(t,x,y)=0$ is considered in [4,5].

Acknowledgments

This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG), Grant # YS-21-554.

References

[1] R. V. Gamkrelidze, *Principles of Optimal Control Theory.* Mathematical Concepts and Methods in Science and Engineering, Vol. 7. Plenum Press, New York–London, 1978.

[2] G. Kharatishvili, T. Tadumadze and N. Gorgodze, Continuous dependence and differentiability of solution with respect to initial data and right-hand side for differential equations with deviating argument. *Mem. Differential Equations Math. Phys.* **19** (2000), 3–105.

[3] I. Ramishvili and T. Tadumadze, Formulas of variation for a solution of neutral differential equations with continuous initial condition. *Georgian Math. J.* **11** (2004), no. 1, 155–175.

[4] T. Tadumadze, Variation formulas of solutions for functional differential equations with several constant delays and their applications in optimal control problems. *Mem. Differ. Equ. Math. Phys.* **70** (2017), 7–97.

[5] T. Tadumadze, Ph. Dvalishvili and T. Shavadze, On the representation of solution of the perturbed controlled differential equation with delay and continuous initial condition. *Appl. Comput. Math.* **18** (2019), no. 3, 305–315.

[6] T. Tadumadze and N. Gorgodze, Variation formulas of a solution and initial data optimization problems for quasi-linear neutral functional differential equations with discontinuous initial condition. *Mem. Differ. Equ. Math. Phys.* **63** (2014), 1–77.