Qualitative Behavior of the Trajectories Impulsive Semigroup for the Hyperbolic Equation

Petro Feketa

Victoria University of Wellington, Wellington, New Zealand
E-mail: petro.feketa@gmail.com

Juliya Fedorenko, Dmytro Bezushchak, Anna Sukretna

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
E-mails: juliamfed@gmail.com; bezushchak@gmail.com; sukretna.a.v@knu.ua

1 Introduction

The study is devoted to an important class of evolutionary systems characterized by the presence of impulsive disturbances when the system trajectory reaches a fixed subset in the phase space. The systematic study of such systems began relatively recently and was mostly focused on the finite-dimensional case [1, 2, 4, 10–12]. The results regarding the limit behavior of infinite-dimensional impulsive dynamic systems are contained in works [3, 6, 8], however, in both the parabolic and hyperbolic cases, the impulsive parameters are "finite-dimensional" in nature, i.e., the situation was considered when only a finite number of coordinates of the phase vector were subjected to an impulsive disturbance. The novelty of this study is that we consider the case when the entire infinite-dimensional phase vector undergoes an impulsive disturbance when the energy functional reaches a certain threshold value.

2 Setting of the problem and the main results

Let a triple of Hilbert spaces $V \subset H \subset V^*$ with compact dense embeddings be given, $\|\cdot\|$ be the norm and $(\cdot,\cdot)$ be the scalar product in H, $A : V \to V^*$ be a linear, continuous, self-adjoint, coercive operator, $\|u\|_V := \langle A^{\frac{1}{2}}u, u\rangle$ be the norm in V, $\langle\cdot,\cdot\rangle$ be the scalar product in V.

Let us consider an evolution problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 2\beta\,\dfrac{dy}{dt} + Ay = 0, \\ y\big|_{t=0} = y_0 \in V, \\ y_t\big|_{t=0} = y_1 \in V. \end{cases} \tag{2.1}$$

Problem (2.1) in phase space $X = V \times H$ generates a continuous semigroup $G : \mathbb{R}_+ \times X \to X$ [13], where for $z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X$

$$G(t, z_0) = z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix} =$$

$$= e^{-\beta t} \sum_{j=1}^{\infty} \begin{pmatrix} (y_0, \varphi_j) \cos \omega_j t + \big(\beta(y_0, \varphi_j) + (y_1, \varphi_j)\big) \dfrac{1}{\omega_j} \sin \omega_j t \\ (y_1, \varphi_j) \cos \omega_j t - \big(\lambda_j^2 (y_0, \varphi_j) + \beta(y_1, \varphi_j)\big) \dfrac{1}{\omega_j} \sin \omega_j t \end{pmatrix}, \quad (2.2)$$

where $\omega_j = \sqrt{\lambda_j^2 - \beta^2}$, $\{\lambda_j\}_{j=1}^{\infty}$, $\{\varphi_j\}_{j=1}^{\infty}$ are solutions of the spectral problem

$$A\varphi_j = \lambda_j \varphi_j, \ \ j \geq 1,$$

$\{\varphi_j\}_{j=1}^{\infty}$ is the orthonormal basis in H, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_j \to +\infty$, $j \to \infty$, and without limitation of generality we will assume that $\lambda_1 > \beta$.

Consider the functional $\Psi : X \to \mathbb{R}_+$, that for $z = \begin{pmatrix} u \\ v \end{pmatrix} \in X$ is determined by the rule

$$\Psi(z) = \|z\|_X^2 = \|u\|_V^2 + \|v\|^2. \tag{2.3}$$

The impulsive problem is formulated as follows: if at some point in time $t > 0$ at the solution $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ the functional (2.3) reaches the value Ψ_0, then the system instantly moves to a new position

$$z^+ = \varphi(z) + \alpha, \tag{2.4}$$

where $\alpha \in X$, $\varphi : X \to X$ are given.

In [9] we prove that, under certain conditions on the parameters, the problem (2.1), (2.3), (2.4) generates in X an impulsive dynamical system $\widetilde{G} : \mathbb{R}_+ \times X \to X$ (see Definition 3.1 below), for which, for each $z_0 \in X$, the ω-boundary set is nonempty, compact, and the limit relation is true

$$\operatorname{dist}_X \big(\widetilde{G}(t, z_0), \widetilde{\omega}(z_0)\big) \to 0, \ \ t \to \infty.$$

3 ω-Boundary set for impulsive dynamical systems

Following the work [7], we will describe the general construction of the impulsive dynamical system. Suppose that a continuous semigroup $G : \mathbb{R}_+ \times X \to X$ is given on the phase space X, the trajectories of semigroup, when they reach a fixed subset $M \subset X$ (impulsive set), are moved by the mapping I (impulsive mapping) to a new position

$$z^+ := Iz.$$

For the correctness of such construction, the following conditions must be met

$$\begin{gathered} G : \mathbb{R}_+ \times X \to X \ \text{ is continuous semigroup,} \\ \text{i.e. for all } z \in X \text{ and } t, s \geq 0\text{: } G(0, z) = z,\ G(t+s, z) = G(t, G(s, z)), \\ \text{map } (t, z) \mapsto G(t, z) \text{ is continuous on } \mathbb{R}_+ \times X; \end{gathered} \tag{3.1}$$

$$M \text{ is closed set}, \ \ M \cap IM = \varnothing; \tag{3.2}$$

$$\forall\, z \in M \ \ \exists\, \tau = \tau(z) > 0 \ \ \forall\, t \in (0, \tau) : \ \ G(t, z) \notin M. \tag{3.3}$$

Under the conditions (3.1)–(3.3) it is known [6] that if for $z \in X$

$$M^+(z) := \Big(\bigcup_{t>0} G(t, z)\Big) \cap M \neq \varnothing,$$

then there exists $\widetilde{s} := \widetilde{s}(z) > 0$ such that

$$\forall\, t \in (0, \widetilde{s}): \quad G(t,z) \notin M, \quad G(\widetilde{s}, z) \in M.$$

Using the introduced notations z^+, $M^+(z)$, $\widetilde{s}$, the impulsive trajectory $\widetilde{G}(\,\cdot\,, z_0)$ starting from $z_0 \in X$ is constructed as follows:

- if $M^+(z_0) = \varnothing$, then $\widetilde{G}(t, z_0) = G(t, z_0)$, $t \geq 0$;

- if $M^+(z_0) \neq \varnothing$, then for $s_0 := \widetilde{s}(z_0)$ let's mark $z_1 := G(s_0, z_0)$, so

$$\widetilde{G}(t, z_0) = \begin{cases} G(t, z_0), & t \in [0, s_0), \\ z_1^+, & t = s_0; \end{cases}.$$

- if $M^+(z_1^+) = \varnothing$, then $\widetilde{G}(t, z_0) = G(t - s_0, z_1^+)$, $t \geq s_0$;

- if $M^+(z_1^+) \neq \varnothing$, then for $s_1 := \widetilde{s}(z_1^+)$ let's mark $z_1 := G(s_1, z_1^+)$, so

$$\widetilde{G}(t, z_0) = \begin{cases} G(t - s_0, z_1^+), & t \in [s_0, s_0 + s_1), \\ z_2^+, & t = s_0 + s_1; \end{cases}$$

and so on. Continuing this process, we will obtain a finite or infinite number of impulsive points

$$z_{n+1}^+ = IG(s_n, z_n^+), \quad z_0^+ := z_0, \quad n \geq 0,$$

and corresponding sequence of time moments

$$T_{n+1} := \sum_{k=0}^{n} s_k, \quad T_0 := 0, \quad n \geq 0.$$

At the same time, $\widetilde{G}$ is given by the formula

$$\widetilde{G}(t, z_0) = \begin{cases} G(t - T_n, z_n^+), & t \in [T_n, T_{n+1}), \\ z_{n+1}^+, & t = T_{n+1}. \end{cases} \tag{3.4}$$

It should be noted that in such a system there may be "beating effects" or "Zeno"-modes, when moments of impulsive occur so often that the trajectory (3.4) is destroyed in a finite time [5].

Since we are interested in the behavior of (3.4) when $t \to \infty$, then we will make the following assumption:

$$\begin{cases} \text{for each } \; z_0 \in X \; \text{ there are either no impulsive points,} \\ \text{or their number is finite, or } \; T_n \to \infty, \;\; n \to \infty. \end{cases} \tag{3.5}$$

The condition (3.5) guarantees that for an arbitrary $z_0 \in X$ the function $t \mapsto \widetilde{G}(t, z_0)$ is defined on $[0, +\infty)$.

Definition 3.1. The mapping $\widetilde{G} : \mathbb{R}_+ \times \mathbb{X} \to X$ constructed above is called an **impulsive dynamic system**. We will say that $\{V, M, I\}$ generate an impulsive dynamic system, if the conditions (3.1)–(3.3), (3.5) are met.

It is known that under the conditions (3.1) – (3.3), (3.5) the mapping $\widetilde{G}:\mathbb{R}_+\times\mathbb{X}\to X$ is a semigroup whose trajectories are continuous from the right.

In addition, by construction for arbitrary $z_0\in X$ and $t>0$:

$$\widetilde{G}(t,z_0)\cap M=\varnothing.$$

The main object of study in this paper is the ω-boundary set:

$$\widetilde{\omega}(z_0)=\Big\{\xi\in X:\ \ \exists\,\{t_n\}_{n=1}^{\infty}:t_n\nearrow\infty,\ \ \xi=\lim_{n\to\infty}\widetilde{G}(t_n,z_0)\Big\}.$$

Lemma 3.1. *Let $\{V,M,I\}$ generate an impulsive dynamic system $\widetilde{G}$ and for $z_0\in X$ the following conditions be fulfilled:*

(1) *set $\widetilde{\gamma}:=\bigcup\limits_{t\geq 0}\widetilde{G}(t,z_0)$ is bounded;*

(2) *for each $z\in\widetilde{\gamma}:G(t,z)=G_1(t,z)+G_2(t,z)$, where $\{G_1(t,z),t\geq 0,z\in\widetilde{\gamma}\}$ is precompact, $\sup\limits_{z\in\widetilde{\gamma}}G_2(t,z)\to 0$, $t\to\infty$.*

(3) *if $\widetilde{\gamma}$ has an infinite number of impulsive points $\{z_n^+\}_{n\geq 0}\}$, then $\{z_n^+\}_{n\geq 0}$ is precompact.*

Then the set $\widetilde{\omega}(z_0)\neq\varnothing$ is compact and $\operatorname{dist}_X(\widetilde{G}(t,z_0),\widetilde{\omega}(z_0))\to 0$*, $t\to\infty$.*

Remark 3.1. Fulfillment of the condition (1) can be guaranteed under the following conditions

$$\begin{gathered}\exists\,C_1,C_2\geq 0\ \ \exists\,\delta>0\ \ \forall\,z\in\widetilde{\gamma}\ \ \forall\,t\geq 0\\ \|G(t,z)\|_X\leq\|z\|_Xe^{-\delta t}+C_1,\\ \|Iz\|_x\leq\|z\|_X+C_2,\end{gathered}$$

and if C$\{s_k\}_{k\geq 0}$ are the distances between impulses along $\widetilde{\gamma}$, then

$$\overline{s}:=\inf_{k\geq 0}s_k>0.$$

Remark 3.2. The condition (3) can be replaced by the following:

$$\text{if } \{z_n\} \text{ is bounded, then } \{Iz_n\} \text{ is precompact.}$$

We cannot expect that $\widetilde{\omega}(z_0)$ to be stable in any sense, since this is not true even in the non-impulsive case. The stability property can be guaranteed for more massive objects – uniform attractors [6]. However, we can ensure the invariance of the non-impulsive part of $\widetilde{\omega}(z_0)$. For this, it is necessary to impose conditions on trajectories starting from initial data close to $\widetilde{\omega}(z_0)$.

Lemma 3.2. *Let $\{V,M,I\}$ generate impulsive dynamical system $\widetilde{G}$, the conditions of Lemma 3.1 be fulfilled for $z_0\in X$, and, in addition*

$$I:M\to X\ \ \textit{be continuous};$$

if $\xi\in\widetilde{\omega}(z_0)\setminus M$, then for $\xi_n\to\xi$

$$\begin{cases}\widetilde{s}(\xi)=\infty, & \textit{if }\ \widetilde{s}(\xi_n)=\infty\ \textit{ for infinitely many } n,\\ \widetilde{s}(\xi_n)\to\widetilde{s}(\xi), & \textit{otherwise.}\end{cases}$$

Then for each $t \geq 0$
$$\widetilde{G}\big(t, \widetilde{\omega}(z_0) \setminus M\big) \subset \widetilde{\omega}(z_0) \setminus M.$$
If in addition for $\xi \in \widetilde{\omega}(z_0) \cap M$ *and for* $\xi_m \to \xi$, $\xi_m \notin M$,
$$\widetilde{s}(\xi_n) = \infty \ \textit{ for infinitely many } n \textit{ or } \widetilde{s}(\xi_n) \to 0,$$
then for arbitrary $t \geq 0$
$$\widetilde{G}\big(t, \widetilde{\omega}(z_0)\big) \supset \widetilde{\omega}(z_0) \setminus M.$$

Remark 3.3. If we add the following condition to the conditions of Lemma 3.2:
$$\text{for } t_n \nearrow \infty \text{ by subsequence } G(t_n, z_0) \to y \notin M, \tag{3.6}$$
then for an arbitrary $t \geq 0$:
$$\widetilde{G}\big(t, \widetilde{\omega}(z_0) \setminus M\big) = \widetilde{\omega}(z_0) \setminus M.$$

The condition (3.6) means that the ω-boundary set of the non-impulsive half-flow G does not intersect with M.

4 Limit modes of the impulsive problem (2.1), (2.3), (2.4)

For the problem (2.1), (2.3), (2.4), the phase space is the Hilbert space $X = V \times H$, on which the solutions of the evolutionary problem (2.1) generate a continuous semigroup $G : \mathbb{R}_+ \times X \to X$ according to the formula (2.2).

The set M is given by (2.3) according to the formula
$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X : \quad \Psi(z) = \Psi_0 \right\}, \quad \Psi_0 > 0.$$

We will consider that the following conditions are fulfilled
$$\|\varphi(z)\|_X \leq \|z\|_X, \quad \Psi_0 < \frac{1}{4}\,\|\alpha\|_X^2. \tag{4.1}$$

In [9] we have checked the fulfillment of the conditions (3.1)–(3.3) and (3.5). Thus, it is proved that the problem (2.1), (2.3), (2.4) generates an impulsive dynamic system, and each impulsive trajectory has an infinite number of impulsive points.

Theorem. *Suppose that for the problem* (2.1), (2.3), (2.4) *the conditions* (4.1) *and the following are fulfilled*
$$\frac{1}{\sqrt{\lambda_1}} < \frac{1}{8\beta}\,\ln\Big(\frac{\|\alpha\|_X^2}{2\Psi_0} - 1\Big), \tag{4.2}$$
$$\varphi : M \to X \ \textit{ is a compact mapping.}$$

Then, for the corresponding impulsive dynamical system $\widetilde{G}$, *we have that for an arbitrary* $z_0 \in X$ ω*-limit set* $\widetilde{\omega}(z_0) \neq \varnothing$, *it is compact and*
$$\operatorname{dist}_X\big(\widetilde{G}(t, z_0), \widetilde{\omega}(z_0)\big) \to 0, \quad t \to \infty.$$

Remark 4.1. The condition (4.2) can be removed by requiring the limit $\lim\limits_{k\to\infty} s_k$ to exist instead.

References

[1] M. U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system. *Nonlinear Anal.* **60** (2005), no. 1, 163–178.

[2] F. Asrorov, V. Sobchuk and O. Kurylko, Finding of bounded solutions to linear impulsive systems. *Eastern-European Journal of Enterprise Technologies* **6** (2019), no. 4(102), 14–20.

[3] E. M. Bonotto, M. C. Bortolan, A. N. Carvalho and R. Czaja, Global attractors for impulsive dynamical systems – a precompact approach. *J. Differential Equations* **259** (2015), no. 7, 2602–2625.

[4] K. Ciesielski, On stability in impulsive dynamical systems. *Bull. Pol. Acad. Sci. Math.* **52** (2004), no. 1, 81–91.

[5] S. Dashkovskiy and P. Feketa, Asymptotic properties of Zeno solutions. *Nonlinear Anal., Hybrid Syst.* **30** (2018), 256–265.

[6] S. Dashkovskiy, P. Feketa, O. V. Kapustyan and I. V. Romaniuk, Existence and invariance of global attractors for impulsive parabolic system without uniqueness. In: Sadovnichiy, V., Zgurovsky, M. (eds) Modern Mathematics and Mechanics. Understanding Complex Systems, pp. 57–78, Springer, Cham, 2019.

[7] S. Dashkovskiy, O. A. Kapustian, O. V. Kapustyan and N. V. Gorban, Attractors for multivalued impulsive systems: existence and applications to reaction-diffusion system. *Mathematical Problems in Engineering* **2021**, Article ID 7385450, 7 pp.

[8] S. Dashkovskiy, O. Kapustyan and I. Romaniuk, Global attractors of impulsive parabolic inclusions. *Discrete & Continuous Dynamical Systems, Series B* **22** (2017), no. 5, 1875–1886.

[9] P. Feketa, J. Fedorenko, D. Bezushchak and A. Sukretna, ω-Limit sets of impulsive semigroups for hyperbolic equations. *Axioms* **2023**, no. 12(10), 918, 14 pp.

[10] V. M. Kirilich, A. D. Myshkis and M. V. Prokhorenko, Oscillations of a membrane under the action of impulsive forces. (Russian) *Ukraïn. Mat. Zh.* **61** (2009), no. 8, 1148–1153; translation in *Ukrainian Math. J.* **61** (2009), no. 8, 1357–1363.

[11] A. M. Samoĭlenko, V. G. Samoĭlenko and V. V. Sobchuk, On periodic solutions of an equation of a nonlinear oscillator with impulse action. (Ukrainian) *Ukraïn. Mat. Zh.* **51** (1999), no. 6, 827–834; translation in *Ukrainian Math. J.* **51** (1999), no. 6, 926–933 (2000).

[12] V. Sobchuk, F. Asrorov, O. Perehuda, A. Sukretna, O. Laptiev and N. Lukova-Chuiko, The limited solutions method for telecommunications network information security models. *Proceedings of the 2021 IEEE 3rd International Conference on Advanced Trends in Information Theory (ATIT)*, pp. 12–131, Kyiv, Ukraine, December 15-18, 2021.

[13] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Second edition. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997.