Asymptotic Behaviour of Solutions of One Class of Nonlinear Differential Equations of Fourth Order

V. M. Evtukhov, S. V. Golubev

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mails: evmod@i.ua; sergii.golubev@stud.onu.edu.ua

We consider a two-membered non-autonomous fourth-order differential equation of the form

\[ y^{(4)} = \alpha_0 p_0(t)[1 + r(t)]e^{\sigma y} \ (\sigma \neq 0), \] (1)

where \( \alpha_0 \in \{-1, 1\} \), \( p_0 : [a, \omega [ \rightarrow ]0, +\infty[ \) is a continuous or continuously differentiable function, \(-\infty < a < \omega \leq +\infty, r : [a, \omega [ \rightarrow ]1, +\infty[ \) is a continuous function such that

\[ \lim_{t \uparrow \omega} r(t) = 0. \]

It is easy to see that in this equation the function \( e^{\sigma y} (\sigma \neq 0) \) is a fast-variable function when \( y \rightarrow Y_0 = \pm \infty \) (by Karamata). We can choose the intervals \( \Delta_{Y_0} \) of the points \( Y_0 = \pm \infty \) as the neighbourhood of \( \Delta_{Y_0} \)

\[ \Delta_{Y_0} = \left\{ \begin{array}{ll} ]0, +\infty[, & \text{if } Y_0 = +\infty, \\ ]-\infty, 0[, & \text{if } Y_0 = -\infty. \end{array} \right. \]

Definition 1. A solution \( y \) of the differential equation (1) is called a \( P_\omega(Y_0, \lambda_0) \)-solution where \(-\infty \leq \lambda_0 \leq +\infty, \) if it is defined on the interval \([t_0, \omega[ \subset [a, \omega[ \) and satisfies the following conditions

\[ y(t) \in \Delta_{Y_0} \text{ or } t \in [t_0, \omega[ \], \lim_{t \uparrow \omega} y(t) = Y_0 = \pm \infty, \]

\[ \lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} 0, & \text{or } 0, \quad (k = 1, 2, 3), \\ \infty, & \text{or } \pm \infty, \end{cases} \quad \lim_{t \uparrow \omega} \frac{y^{(3)}(t)^2}{y^{(2)}(t)y^{(4)}(t)} = \lambda_0. \]

From this definition, in particular, it follows that the number of

\[ \nu_0 = \begin{cases} 1, & \text{or } Y_0 = +\infty, \\ -1, & \text{or } Y_0 = -\infty \end{cases} \]

determines the signs of any \( P_\omega(Y_0, \lambda_0) \)-solution and its first derivative in any left neighbourhood of \( \omega \). In [1] for \( P_\omega(Y_0, \lambda_0) \)-solutions at \( \lambda_0 \in R \setminus \{0, \frac{1}{2}, \frac{2}{3}, 1\} \) (non special case) the following two theorems were obtained, but to formulate them we need to introduce additional auxiliary notations

\[ K(\lambda_0) = \frac{(\lambda_0 - 1)^3}{\lambda_0(2\lambda_0 - 1)}, \quad \pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases} \]

\[ J_0(t) = \int_{\lambda_0}^{t} \pi_\omega(\tau)p_0(\tau) d\tau, \quad J_1(t) = \int_{\lambda_0}^{t} p_0(\tau) J_0(\tau) d\tau, \quad J_i(t) = \int_{\lambda_0}^{t} J_{i-1}(\tau) d\tau (i = 2, 3), \]

\[ Y(t) = -\frac{1}{\sigma} \ln \left( \alpha_0 \left( -\frac{1}{\sigma} \right) K(\lambda_0) J_0(t) \right), \quad q(t) = \frac{Y'(t)}{\alpha_0 J_3(t)}, \]
where the integration boundary $A_i$ is chosen to be equal to either $\omega$ or constant $a$ and is defined in such a way that at this value of $A_i$ the integral tends either to 0 or to $\pm \infty$. The following two theorems were established for equation (1) in [1].

**Theorem 1.** Let $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}, 1\}$. For the differential equation (1) to have $P_\omega(Y_0, \lambda_0)$-solutions, the following inequalities

$$\alpha_0 \nu_0 \lambda_0 (2\lambda_0 - 1)(3\lambda_0 - 2) > 0, \quad \alpha_0 \nu_1 K(\lambda_0) \pi_\omega(t) > 0 \text{ at } t \in ]a, \omega[,$$

(2)

and the following conditions

$$\alpha_0 \sigma K(\lambda_0) J_0(t) < 0 \text{ at } t \in ]a, \omega[,$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_0'(t)}{J_0(t)} = \pm \infty, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_1'(t)}{J_1(t)} = \frac{1}{\lambda_0 - 1}, \quad \lim_{t \uparrow \omega} q(t) = 1$$

(3)

must be satisfied and each such solution admits at $t \uparrow \omega$ the following asymptotic mappings

$$y(t) = -\frac{1}{\sigma} \ln \left( \alpha_0 \left( -\frac{1}{\sigma} \right) K(\lambda_0) J_0(t) \right) + o(1), \quad y^{(k)}(t) = \alpha_0 J_{4-k}(1 + o(1)) \quad (k = 1, 2, 3).$$

**Theorem 2.** Let $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}, 1\}$, the function $p_0$ be continuous and conditions (2), (3) be satisfied. Let, in addition

$$\lim_{t \uparrow \omega} (1 - q(t))|Y(t)|^{\frac{3}{2}} = 0 \text{ and } \alpha_0 \sigma > 0.$$  

(4)

Then the differential equation (1) has a two-parameter family $P_\omega(Y_0, \lambda_0)$ of solutions which satisfy at $t \uparrow \omega$ the asymptotic mappings

$$y(t) = Y(t) + o(1), \quad y'(t) = \alpha_0 J_3(t) \left[ 1 + \frac{o(1)}{|Y(t)|^{\frac{3}{2}}} \right], \quad y''(t) = \alpha_0 J_2(t) \left[ 1 + \frac{o(1)}{|Y(t)|^{\frac{3}{2}}} \right],$$

$$y'''(t) = \alpha_0 J_1(t) \left[ 1 + \frac{o(1)}{|Y(t)|^{\frac{3}{2}}} \right].$$

In Theorem 2, the first of conditions (4) is rather rigid. In the present paper an attempt is made to eliminate it.

**Theorem 3.** Let $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}, 1\}$, the function $p_0$ be continuously differentiable and conditions (2), (3) be satisfied. Suppose, in addition, that the second condition in (4) is satisfied and there exists a finite or equal to $\pm \infty$ limit

$$\lim_{t \uparrow \omega} \pi_\omega(t) q'(t).$$

Then the differential equation (1) has a two-parameter family $P_\omega(Y_0, \lambda_0)$ of solutions which satisfy at $t \uparrow \omega$ the asymptotic mappings

$$y(t) = Y(t) + o(1), \quad y'(t) = \alpha_0 J_3(t) \left[ q(t) + \frac{o(1)}{|Y(t)|^{\frac{3}{2}}} \right], \quad y''(t) = \alpha_0 J_2(t) \left[ 1 + \frac{o(1)}{|Y(t)|^{\frac{3}{2}}} \right],$$

$$y'''(t) = \alpha_0 J_1(t) \left[ 1 + \frac{o(1)}{|Y(t)|^{\frac{3}{2}}} \right].$$

(5)
Sketch of the proof

First, it is easy to prove that
\[ \lim_{t \uparrow \omega} \pi_\omega(t) q'(t) = 0. \]

In the same way as in the proof of Theorem 2 of [1], equation (1) by the transformation
\[ y(t) = Y(t) + y_1(t), \quad y^{(k)}(t) = \alpha_0 J_{4-k}(t)[1 + y_{k+1}(t)] \quad (k = 1, 2, 3) \]  
(6)
is reduced to a system of differential equations of the form
\begin{align*}
y_1' &= \alpha_0 J_3(t)[1 - q(t) + y_2], \\
y_2' &= \frac{J_2(t)}{J_3(t)} (y_3 - y_2), \\
y_3' &= \frac{J_2(t)}{J_1(t)} (y_4 - y_3), \\
y_4' &= \frac{J_1(t)}{J_1(t)} \left[ r(t) + (1 + r(t))y_1 - y_4 + R(t, y_1) \right].
\end{align*}

We will consider this system on the set
\[ \Omega = [t_1, \omega[ \times D, \quad \text{where} \quad D = \left\{ (y_1, y_2, y_3, y_4) \in \mathbb{R}^4_+ : |y_i| \leq \frac{1}{2}, \ (i = 1, \ldots, 4) \right\}, \]
where \( |R(t, y_1)| \leq y_1^2 \) at \( |y_1| \leq \delta \) for some \( 0 < \delta < \frac{1}{2} \).

Further we will use the obtained system on the set \( \Omega_0 = [t_1, \omega[ \times \mathbb{R}^4_+ \).

In contrast to Theorem 2, let us make an additional transformation
\[ y_1(t) = z_1(t), \quad y_2(t) = z_2(t) + q(t) - 1, \quad y_3(t) = z_3(t), \quad y_4(t) = z_4(t), \]
(7)
the sense of which is to exclude the summand \((1 - q(t))\) from the first equation of the system and as a result we obtain a system of differential equations of the form
\begin{align*}
z_1' &= \frac{Y(t)}{\pi_\omega(t)} \left\{ \xi_1(t)z_2 \right\}, \\
z_2' &= \frac{1}{\pi_\omega(t)} \left\{ \xi_2(t)(z_3 - z_2) - \pi_\omega(t)q'(t) \right\}, \\
z_3' &= \frac{1}{\pi_\omega(t)} \left\{ \xi_3(t)(z_4 - z_3) \right\}, \\
z_4' &= \frac{1}{\pi_\omega(t)} \left\{ \xi_4(t)[r(t) + (1 + r(t))z_1 - z_4 + R(t, z_1)] \right\},
\end{align*}
(8)
where
\begin{align*}
\lim_{t \uparrow \omega} \xi_1(t) &= \frac{3\lambda_0 - 2}{\lambda_0 - 1}, \\
\lim_{t \uparrow \omega} \xi_2(t) &= \frac{2\lambda_0 - 1}{\lambda_0 - 1}, \\
\lim_{t \uparrow \omega} \xi_3(t) &= \frac{\lambda_0}{\lambda_0 - 1}, \\
\lim_{t \uparrow \omega} \xi_4(t) &= \frac{1}{\lambda_0 - 1}.
\end{align*}

To asymptotically equalise the multipliers at \( t \uparrow \omega \) in the right-hand side of the equations of the system (8), we apply the following transformation to it:
\[ z_1(t) = v_1(t), \quad z_2(t) = |Y(t)|^{-\frac{1}{2}} v_2(t), \quad z_3(t) = |Y(t)|^{-\frac{1}{2}} v_3(t), \quad z_4(t) = |Y(t)|^{-\frac{1}{2}} v_4(t). \]  
(9)
As a result, we obtain a system of quasilinear differential equations for which all the conditions of Theorem 2.2 of [2] are fulfilled. The limit matrix of coefficients at \( v_1, v_2, v_3, v_4 \) of the obtained quasilinear system has the form

\[
C = \begin{pmatrix}
0 & \frac{3\lambda_0 - 2}{\lambda_0 - 1} \left( \frac{v_0}{\text{sign} \sigma} \right) & 0 & 0 \\
0 & 2\lambda_0 - 1 & 0 & 0 \\
0 & 0 & \lambda_0 & \frac{\lambda_0}{\lambda_0 - 1} \\
1 & 0 & 0 & 0 \\
\end{pmatrix},
\]

and has, taking into account the sign conditions (2), (3), a characteristic equation of the form

\[
\lambda^4 + \frac{\alpha_0}{\sigma} \frac{|3\lambda_0 - 2| |2\lambda_0 - 1| |\lambda_0|}{(\lambda_0 - 1)^4} = 0.
\]

The characteristic equation has two pairs of complex-conjugate roots with real parts different from zero. Then the system of differential equations has a two-parameter family of solutions \( v_1, v_2, v_3, v_4 : [t_2, \omega[ \to \mathbb{R}^4 \ (t_2 \in [t_0, \omega]), which tend to 0 at \ t \uparrow \omega. To each such solution, taking into account substitutions (6), (7), (9), corresponds a solution \( y : [t_2, \omega[ \to \mathbb{R} of the differential equation (1) for which the asymptotic representations (5) take place at \ t \uparrow \omega. It is also easy to check, taking into account these asymptotic representations (5) take place at \ t \uparrow \omega. It is also easy to check, taking into account these asymptotic representations and the form of equation (1), that the solutions we have constructed are \( P_\omega(Y_0, \lambda_0)\)-solutions.

References
