Asymptotic Behaviour of Solutions of One Class of Nonlinear Differential Equations of Fourth Order

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We consider a two-membered non-autonomous fourth-order differential equation of the form

$$y^{(4)} = \alpha_0 p_0(t)[1+r(t)]e^{\sigma y} \ \ (\sigma \neq 0), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p_0 : [a,\omega[\to]0,+\infty[$ is a continuous or continuously differentiable function, $-\infty < a < \omega \leq +\infty$, $r : [a,\omega[\to]-1,+\infty[$ is a continuous function such that

$$\lim_{t\uparrow\omega} r(t) = 0.$$

It is easy to see that in this equation the function $e^{\sigma y}$ $(\sigma \neq 0)$ is a fast-variable function when $y \to Y_0 = \pm\infty$ (by Karamata). We can choose the intervals Δ_{Y_0} of the points $Y_0 = \pm\infty$ as the neighbourhood of Δ_{Y_0}

$$\Delta_{Y_0} = \left[\begin{array}{ll}]0,+\infty[\,, & \text{if } Y_0 = +\infty, \\ ]-\infty,0[\,, & \text{if } Y_0 = -\infty. \end{array}\right.$$

Definition 1. A solution y of the differential equation (1) is called a $P_\omega(Y_0,\lambda_0)$-solution where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on the interval $[t_0,\omega[\subset [a,\omega[$ and satisfies the following conditions

$$y(t) \in \Delta_{Y_0} \ \text{ or } \ t \in [t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y(t) = Y_0 = \pm\infty,$$

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \left[\begin{array}{l} \text{or } \ 0, \\ \text{or } \ \pm\infty, \end{array}\right. \quad (k=1,2,3), \qquad \lim_{t\uparrow\omega} \frac{[y^{(3)}(t)]^2}{y^{(2)}(t)y^{(4)}(t)} = \lambda_0.$$

From this definition, in particular, it follows that the number of

$$\nu_0 = \begin{cases} 1, & \text{or } \ Y_0 = +\infty, \\ -1, & \text{or } \ Y_0 = -\infty \end{cases}$$

determines the signs of any $P_\omega(Y_0,\lambda_0)$-solution and its first derivative in any left neighbourhood of ω. In [1] for $P_\omega(Y_0,\lambda_0)$-solutions at $\lambda_0 \in \mathbb{R} \setminus \{0,\frac{1}{2},\frac{2}{3},1\}$ (non special case) the following two theorems were obtained, but to formulate them we need to introduce additional auxiliary notations

$$K(\lambda_0) = \frac{(\lambda_0-1)^3}{\lambda_0(2\lambda_0-1)}\,, \quad \pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t-\omega, & \text{if } \ \omega < +\infty, \end{cases}$$

$$J_0(t) = \int_{A_0}^{t} \pi_\omega^3(\tau)p_0(\tau)\,d\tau, \quad J_1(t) = \int_{A_1}^{t} \frac{p_0(\tau)}{J_0(\tau)}\,d\tau, \quad J_i(t) = \int_{A_i}^{t} J_{i-1}(\tau)\,d\tau \ (i=2,3),$$

$$Y(t) = -\frac{1}{\sigma}\ \ln\Big(\alpha_0\Big(-\frac{1}{\sigma}\Big)K(\lambda_0)J_0(t)\Big), \quad q(t) = \frac{Y'(t)}{\alpha_0 J_3(t)}\,,$$

where the integration boundary A_i is chosen to be equal to either ω or constant a and is defined in such a way that at this value of A_i the integral tends either to 0 or to $\pm\infty$. The following two theorems were established for equation (1) in [1].

Theorem 1. *Let $\lambda_0 \in \mathbb{R}\setminus\{0, \frac{1}{2}, \frac{2}{3}, 1\}$. For the differential equation* (1) *to have $P_\omega(Y_0, \lambda_0)$-solutions, the following inequalities*

$$\alpha_0\nu_0\lambda_0(2\lambda_0-1)(3\lambda_0-2)>0, \quad \alpha_0\nu_1 K(\lambda_0)\pi_\omega(t)>0 \ \ at \ \ t\in\,]a,\omega[\,, \tag{2}$$

and the following conditions

$$\begin{gathered}\alpha_0\sigma K(\lambda_0)J_0(t)<0 \ \ at \ \ t\in\,]a,\omega[\,,\\ \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_0'(t)}{J_0(t)}=\pm\infty, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_1'(t)}{J_1(t)}=\frac{1}{\lambda_0-1}\,, \quad \lim_{t\uparrow\omega}q(t)=1\end{gathered} \tag{3}$$

must be satisfied and each such solution admits at $t\uparrow\omega$ the following asymptotic mappings

$$y(t)=-\frac{1}{\sigma}\,\ln\Big(\alpha_0\Big(-\frac{1}{\sigma}\Big)K(\lambda_0)J_0(t)\Big)+o(1), \quad y^{(k)}(t)=\alpha_0J_{4-k}(t)[1+o(1)] \ \ (k=1,2,3).$$

Theorem 2. *Let $\lambda_0 \in \mathbb{R}\setminus\{0, \frac{1}{2}, \frac{2}{3}, 1\}$, the function p_0 be continuous and conditions* (2), (3) *be satisfied. Let, in addition*

$$\lim_{t\uparrow\omega}(1-q(t))|Y(t)|^{\frac{3}{4}}=0 \ \ and \ \ \alpha_0\sigma>0. \tag{4}$$

Then the differential equation (1) *has a two-parameter family $P_\omega(Y_0, \lambda_0)$ of solutions which satisfy at $t\uparrow\omega$ the asymptotic mappings*

$$\begin{gathered}y(t)=Y(t)+o(1), \quad y'(t)=\alpha_0J_3(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{3}{4}}}\Big], \quad y''(t)=\alpha_0J_2(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{2}}}\Big],\\ y'''(t)=\alpha_0J_1(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{4}}}\Big].\end{gathered}$$

In Theorem 2, the first of conditions (4) is rather rigid. In the present paper an attempt is made to eliminate it.

Theorem 3. *Let $\lambda_0 \in \mathbb{R}\setminus\{0, \frac{1}{2}, \frac{2}{3}, 1\}$, the function p_0 be continuously differentiable and conditions* (2), (3) *be satisfied. Suppose, in addition, that the second condition in* (4) *is satisfied and there exists a finite or equal to $\pm\infty$ limit*

$$\lim_{t\uparrow\omega}\pi_\omega(t)q'(t).$$

Then the differential equation (1) *has a two-parameter family $P_\omega(Y_0, \lambda_0)$ of solutions which satisfy at $t\uparrow\omega$ the asymptotic mappings*

$$\begin{gathered}y(t)=Y(t)+o(1), \quad y'(t)=\alpha_0J_3(t)\Big[q(t)+\frac{o(1)}{|Y(t)|^{\frac{3}{4}}}\Big], \quad y''(t)=\alpha_0J_2(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{2}}}\Big],\\ y'''(t)=\alpha_0J_1(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{4}}}\Big].\end{gathered} \tag{5}$$

Sketch of the proof

First, it is easy to prove that
$$\lim_{t\uparrow\omega}\pi_\omega(t)q'(t)=0.$$
In the same way as in the proof of Theorem 2 of [1], equation (1) by the transformation
$$y(t)=Y(t)+y_1(t),\quad y^{(k)}(t)=\alpha_0 J_{4-k}(t)[1+y_{k+1}(t)]\ \ (k=1,2,3) \tag{6}$$
is reduced to a system of differential equations of the form
$$\begin{aligned}
y_1'&=\alpha_0 J_3(t)\big[1-q(t)+y_2\big],\\
y_2'&=\frac{J_3'(t)}{J_3(t)}\,(y_3-y_2),\\
y_3\prime&=\frac{J_2'(t)}{J_2(t)}\,(y_4-y_3),\\
y_4'&=\frac{J_1'(t)}{J_1(t)}\,\big[r(t)+(1+r(t))y_1-y_4+R(t,y_1)\big].
\end{aligned}$$
We will consider this system on the set
$$\Omega=[t_1,\omega[\times D,\ \text{ where }\ D=\Big\{(y_1,y_2,y_3,y_4)\in\mathbb{R}^4_{\frac12}:\ \ |y_i|\le\frac12,\ \ (i=1,\dots,4)\Big\},$$
$$\text{where }\ |R(t,y_1|\leqslant y_1^2\ \text{ at }\ |y_1|\leqslant\delta\ \text{ for some }\ 0<\delta<\frac12\,.$$
Further we will use the obtained system on the set $\Omega_0=[t_1,\omega[\,\times\mathbb{R}^4_\delta$.

In contrast to Theorem 2, let us make an additional transformation
$$y_1(t)=z_1(t),\quad y_2(t)=z_2(t)+q(t)-1,\quad y_3(t)=z_3(t),\quad y_4(t)=z_4(t), \tag{7}$$
the sense of which is to exclude the summand $(1-q(t))$ from the first equation of the system and as a result we obtain a system of differential equations of the form
$$\begin{aligned}
z_1'&=\frac{Y(t)}{\pi_\omega(t)}\,\big\{\xi_1(t)z_2\big\},\\
z_2'&=\frac{1}{\pi_\omega(t)}\,\big\{\xi_2(t)(z_3-z_2)-\pi_\omega(t)q'(t)\big\},\\
z_3'&=\frac{1}{\pi_\omega(t)}\,\big\{\xi_3(t)(z_4-z_3)\big\},\\
z_4'&=\frac{1}{\pi_\omega(t)}\,\Big\{\xi_4(t)\big[r(t)+(1+r(t))z_1-z_4+R(t,z_1)\big]\Big\},
\end{aligned}\tag{8}$$
where
$$\lim_{t\uparrow\omega}\xi_1(t)=\frac{3\lambda_0-2}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}\xi_2(t)=\frac{2\lambda_0-1}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}\xi_3(t)=\frac{\lambda_0}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}\xi_4(t)=\frac{1}{\lambda_0-1}\,.$$
To asymptotically equalise the multipliers at $t\uparrow\omega$ in the right-hand side of the equations of the system (8), we apply the following transformation to it:
$$z_1(t)=\upsilon_1(t),\quad z_2(t)=|Y(t)|^{-\frac34}\upsilon_2(t),\quad z_3(t)=|Y(t)|^{-\frac12}\upsilon_3(t),\quad z_4(t)=|Y(t)|^{-\frac14}\upsilon_4(t). \tag{9}$$

As a result, we obtain a system of quasilinear differential equations for which all the conditions of Theorem 2.2 of [2] are fulfilled. The limit matrix of coefficients at υ_1, υ_2, υ_3, υ_4 of the obtained quasilinear system has the form

$$C = \begin{pmatrix} 0 & \dfrac{3\lambda_0 - 2}{\lambda_0 - 1}\Big(\dfrac{\nu_0}{\operatorname{sign}\sigma}\Big) & 0 & 0 \\ 0 & 0 & \dfrac{2\lambda_0 - 1}{\lambda_0 - 1} & 0 \\ 0 & 0 & 0 & \dfrac{\lambda_0}{\lambda_0 - 1} \\ \dfrac{1}{\lambda_0 - 1} & 0 & 0 & 0 \end{pmatrix},$$

and has, taking into account the sign conditions (2), (3), a characteristic equation of the form

$$\lambda^4 + \frac{\alpha_0}{\sigma}\,\frac{|3\lambda_0 - 2|\,|2\lambda_0 - 1|\,|\lambda_0|}{(\lambda_0 - 1)^4} = 0.$$

The characteristic equation has two pairs of complex-conjugate roots with real parts different from zero. Then the system of differential equations has a two-parameter family of solutions $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 : [t_2, \omega[\to \mathbb{R}^4_\delta$ ($t_2 \in [t_0, \omega[$), which tend to 0 at $t \uparrow \omega$. To each such solution, taking into account substitutions (6), (7), (9), corresponds a solution $y : [t_2, \omega[\to \mathbb{R}$ of the differential equation (1) for which the asymptotic representations (5) take place at $t \uparrow \omega$. It is also easy to check, taking into account these asymptotic representations and the form of equation (1), that the solutions we have constructed are $P_\omega(Y_0, \lambda_0)$-solutions.

References

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[2] V. M. Evtukhov and A. M. Samoilenko, Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point. (Russian) *Ukr. Mat. Zh.* **62** (2010), no. 1, 52–80; translation in *Ukr. Math. J.* **62** (2010), no. 1, 56–86.