On the Optimization Problem of One Market Relation Containing the Delay Functional Differential Equation

Phridon Dvalishvili

Department of Computer Sciences, Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia
E-mail: pridon.dvalishvili@tsu.ge

Adeljalil Nachaoui

Laboratoire de Mathématiques Jean Leray, Nantes Université, Nantes, France
E-mail: abdeljalil.nachaoui@univ-nantes.fr

Mourad Nachaoui

Université Sultan Moulay Slimane, Béni-Mellal, Morocco
E-mail: m.nachaoui@usms.ma

Tamaz Tadumadze[1,2]

[1]*Department of Mathematics, Ivane Javakhishvili Tbilisi State University*
Tbilisi, Georgia
[2]*Ilia Vekua Institute of Applied Mathematics of Ivane Javakhishvili Tbilisi State University*
Tbilisi, Georgia
E-mail: tamaz.tadumadze@tsu.ge

In the paper, for a market relation theoretical model is constructed in the form of the controlled delay functional differential equation. Moreover, for the corresponding optimization problem the necessary conditions of optimality are formulated.

1 Mathematical model

Let us for the production of goods i_1 and i_2 require substitutable raw materials with concentration $x_1(t)$ and $x_2(t)$, respectively, at the moment t. Let the dynamic of these concentrations is described by the system of differential equations

$$\begin{cases} \dot{x}_1(t) = ax_1(t) + bx_2(t), \\ \dot{x}_2(t) = cx_1(t) + dx_2(t), \end{cases}$$

where a, b, c, d are given numbers.

Let market relation demand and supply for the good i_1 are described by functions $D_1(t,\omega)$ and $S_1(t,x_1,x_2,u)$ and for the good i_2 are described by functions $D_2(t,\vartheta)$ and $S_2(t,x_1,x_2,v)$. Let cost of the goods i_1 and i_2 at the moment t be $u(t)$ and $v(t)$, respectively. Suppose that at time t consumer demand will be satisfied on the good i_1 which has been ordered at time $t-\rho$, where $\rho>0$ is a fixed delay parameter and on the good i_2 which has been ordered at time $t-\theta$, where $\theta>0$, in general, is non fixed delay. The function

$$E_1(t) = D_1(t-\rho, u(t-\rho)) - S_1\big(t, x_1(t-\tau), x_2(t-\tau), u(t)\big), \ \ t \in I$$

we call the disbalance index for the good i_1. We assume that for the production of the good i_1 requires the amount of raw materials $x_1(t-\tau)$ and $x_2(t-\tau)$ allocated at moments $t-\tau$, where $\tau > 0$, in general, is non fixed delay. Here, it is taken into account that the production of i_1 good is carried out after some time from the allocation of raw materials.

Similarly, the function

$$E_2(t) = D_2(t-\theta, v(t-\theta)) - S_2\big(t, x_1(t-\tau), x_2(t-\tau), v(t)\big), \quad t \in I$$

is called the disbalance index for the good i_2.

If $E_1(t) = 0$, then at the moment t we do not have disbalance between demand and supply with respect to good i_1, and the customer will buy exactly the quantity of good i_1 he needs. At time t, if $E_1(t) > 0$, then demand exaggerates supply, if $E_1(t) < 0$, then supply exaggerates demand. Analogously we can consider above described cases for $E_2(t)$.

In order to characterize the dynamics of the disbalance in time, we introduce the integral indices of the disbalance for the moment t

$$x_3(t) = x_{30} + \int_{t_0}^{t} \Big[D_1(\xi-\rho, u(\xi-\rho)) - S_1\big(\xi, x_1(\xi-\tau), x_2(\xi-\tau), u(\xi)\big)\Big]\, d\xi,$$

$$x_4(t) = x_{40} + \int_{t_0}^{t} \Big[D_2(\xi-\theta, v(\xi-\theta)) - S_2\big(\xi, x_1(\xi-\tau), x^2(\xi-\tau), v(\xi)\big)\Big]\, d\xi,$$

where x_{i0}, $i = 3, 4$ are given numbers. Thus, in the framework of the above mentioned conditions, we can describe the market relationship with the following system of controlled functional differential equation containing delays in phase coordinates and controls

$$\begin{cases} \dot{x}_1(t) = ax_1(t) + bx_2(t), \\ \dot{x}_2(t) = cx_1(t) + dx_2(t), \\ \dot{x}_3(t) = D_1(t-\rho, u(t-\rho)) - S_1(t, x_1(t-\tau), x_2(t-\tau), u(t)), \\ \dot{x}_4(t) = D_2(t-\theta, v(t-\theta)) - S_2(t, x_1(t-\tau), x_2(t-\tau), v(t)). \end{cases} \tag{1.1}$$

Finally, we note that one dimensional models for the market relation and corresponding optimization problems when right-hand side of the differential equation depends only on control both without delay and with delay were discussed in [1–4].

2 Statement of the problem and necessary conditions of optimality

Let $I = [t_0, t_1]$ be a given interval and $\tau_2 > \tau_1 > 0$, $\rho > 0$ and $\theta_2 > \theta_1 > 0$ be given numbers with $t_1 - t_0 > \max\{\tau_2, \rho, \theta_2\}$. Suppose that the functions $\varphi_i(t) \in R_+ = (0, \infty)$, $i = 1, 2$ are continuously differentiable on the interval $[\widehat{\tau}, t_0]$, where $\widehat{\tau} = t_0 - \tau_2$. Further, denote by Ω and V the set of piecewise continuous control functions $u(t) \in [0, \widehat{u}]$, $t \in [t_0 - \rho, t_1]$ and continuously differentiable control functions $v(t) \in [0, \widehat{v}]$, $t \in [t_0 - \theta_2, t_1]$, respectively, where $\widehat{u} > 0$, $\widehat{v} > 0$ are given numbers.

To each element $w = (\tau, \theta, u(t), v(t)) \in W = (\tau_1, \tau_2) \times (\theta_1, \theta_2) \times \Omega \times V$ we assign the delay functional differential equation (1.1) with the initial condition

$$x_i(t) = \varphi_i(t), t \in [\widehat{\tau}, t_0), x_i(t_0) = x_{i0}, i = 1, 2; x_i(t_0) = x_{i0}, i = 3, 4, \tag{2.1}$$

where $x_{i0} \in R_+$, $i = 1, 2$ and, in general, $\varphi_i(t_0) \neq x_{i0}$, $i = 1, 2$ (so called discontinuous part of the condition (2.1)). In the equation (1.1) it is assumed that the function $D_1(t, \omega), (t, \omega) \in [t_0 - \rho, t_1] \times [0, \widehat{u}]$ is continuous and continuously differentiable with respect to ω; the function $S_1(t, x_1, x_2, u)$, $(t, x_1, x_2, u) \in I \times R_+^2 \times [0, \widehat{u}]$ is continuous and continuously differentiable with respect to x_1, x_2, u; the function $D_2(t, \vartheta)$, $(t, \vartheta) \in [t_0 - \theta_2, t_1] \times [0, \widehat{v}]$ is continuous and continuously differentiable with respect to ϑ; the function $S_2(t, x_1, x_2, v)$, $(t, x_1, x_2, v) \in I \times R_+^2 \times [0, \widehat{v}]$ is continuous and continuously differentiable with respect to x_1, x_2, v.

Definition 1. Let $w = (\tau, \theta, u(t), v(t)) \in W$. The collection of functions

$$\Big\{x_i(t) = x_i(t; w) \in R_+, \ t \in [\widehat{\tau}, t_1], \ i = 1, 2; \ \ x_i(t) = x_i(t; w), \ t \in I, \ i = 3, 4\Big\}$$

is called a solution of the equation (1.1) with the initial condition (2.1) or a solution corresponding to the element w, if it satisfies the condition (2.1) and the functions $x_i(t)$, $i = 1, 2, 3, 4$ are absolutely continuous on the interval I and satisfy the equation (1.1) almost everywhere on I.

Denote by W_0 the set of $w \in W$ for which there exists a solution. We assume that $W_0 \neq \varnothing$.

Definition 2. An element $w_0 = (\tau_0, \theta_0, u_0(t), v_0(t)) \in W_0$ is said to be optimal if for an arbitrary element $w \in W_0$ the inequality

$$J(w_0) \leq J(w)$$

holds, where

$$J(w) = \int\limits_{t_0}^{t_1} \big[x_3^2(t) + x_4^2(t)\big]\, dt$$

and $x_i(t) = x_i(t; w)$, $i = 3, 4$.

Theorem 1. *Let w_0 be an optimal element and*

$$\Big\{x_{i0}(t) = x_i(t; w_0) \in R_+, \ t \in [\widehat{\tau}, t_1], \ i = 1, 2; \ \ x_{i0}(t) = x_i(t; w_0), \ t \in I, \ i = 3, 4\Big\}$$

be the corresponding solution. Let the function $u_0(t)$ be continuous at the point $t_0 + \tau_0$. Then there exists a solution $\{\psi_i(t), \ t \in [t_0, t_1 + \tau_0], \ i = 1, 2, 3, 4\}$ of the equation

$$\begin{cases}
\dot{\psi}_1(t) = -a\psi_1(t) - c\psi_2(t) + S_{1x_1}[t + \tau_0]\psi_3(t + \tau_0) + S_{2x_1}[t + \tau_0]\psi_4(t + \tau_0),\\
\dot{\psi}_2(t) = -b\psi_1(t) - d\psi_2(t) + S_{1x_2}[t + \tau_0]\psi_3(t + \tau_0) + S_{2x_2}[t + \tau_0]\psi_4(t + \tau_0),\\
\dot{\psi}_3(t) = 2x_{30}(t),\\
\dot{\psi}_4(t) = 2x_{40}(t),\\
t \in I
\end{cases}$$

with the initial condition

$$\psi_i(t) = 0, \ \ t \in [t_1, t_1 + \tau_0], \ \ i = 1, 2, 3, 4,$$

where

$$S_{1x_1}[t] = \frac{\partial}{\partial x_1}\, S_1\big(t, x_{10}(t - \tau_0), x_{20}(t - \tau_0), u_0(t)\big),$$
$$S_{2x_1}[t] = \frac{\partial}{\partial x_1}\, S_2\big(t, x_{10}(t - \tau_0), x_{20}(t - \tau_0), v_0(t)\big)$$

such that the following conditions hold:

1) *the condition for the delay* τ_0

$$\widehat{S}_1\psi_3(t_0+\tau_0)+\widehat{S}_2\psi_4(t_0+\tau_0)$$
$$=\int\limits_{t_0}^{t_1}\Big\{\big[S_{1x_1}[t]\psi_3(t)+S_{2x_1}[t]\psi_4(t)\big]\dot{x}_{10}(t-\tau_0)+\big[S_{1x_2}[t]\psi_3(t)+S_{2x_2}[t]\psi_4(t)\big]\dot{x}_{20}(t-\tau_0)\Big\}\,dt,$$

where

$$\widehat{S}_1=S_1\big(t_0+\tau_0,\varphi_1(t_0),\varphi_2(t_0),u_0(t_0+\tau_0)\big)-S_1\big(t_0+\tau_0,x_{10},x_{20},u_0(t_0+\tau_0)\big),$$
$$\widehat{S}_2=S_2\big(t_0+\tau_0,\varphi_1(t_0),\varphi_2(t_0),v_0(t_0+\tau_0)\big)-S_2\big(t_0+\tau_0,x_{10},x_{20},v_0(t_0+\tau_0)\big);$$

2) *the condition for the delay* θ_0

$$\int\limits_{t_0}^{t_1}D_{2\vartheta}(t-\theta_0,v_0(t-\theta_0))\psi_4(t)\dot{v}_0(t-\theta_0)\,dt=0;$$

3) *the condition for the control* $u_0(t)$

$$\int\limits_{t_0}^{t_1}\Big\{\psi_3(t)\big[-S_{1u}[t]u_0(t)+D_{1\omega}(t-\rho,u_0(t-\rho))u_0(t-\rho)\big]\Big\}\,dt$$
$$=\max_{u(t)\in\Omega}\int\limits_{t_0}^{t_1}\Big\{\psi_3(t)\big[-S_{1u}[t]u(t)+D_{1\omega}(t-\rho,u_0(t-\rho))u(t-\rho)\big]\Big\}\,dt;$$

4) *the condition for the control* $v_0(t)$

$$\int\limits_{t_0}^{t_1}\Big\{\psi_4(t)\big[-S_{2v}[t]v_0(t)+D_{2w}(t-\theta_0,v_0(t-\theta_0))v_0(t-\theta_0)\big]\Big\}\,dt$$
$$=\max_{v(t)\in V}\int\limits_{t_0}^{t_1}\Big\{\psi_4(t)\big[-S_{2v}[t]v(t)+D_{2w}(t-\theta_0,v_0(t-\theta_0))v(t-\theta_0)\big]\Big\}\,dt.$$

It is clear that if $\varphi_i(t_0)=x_{i0}$, $i=1,2$, then $\widehat{S}_i=0$, $i=1,2$. Theorem 1 is proved by the scheme given in [5].

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