On the Massera's Theorem of Existence of Periodic Solutions of Linear Differential Systems

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Many areas of modern physics and technology are substantially based on various oscillatory processes or use them. Oscillatory processes also play an important, and sometimes determining role in a significant part of natural phenomena. These circumstances determine the relevance of research in the oscillation theory and the necessity for its development. Although an effective apparatus for studying oscillations in nonlinear systems is developed in modern oscillation theory, the "linear" part of the theory remains an important and demanded part of it both in theoretical and practical aspects. At the same time, the center of gravity of practical research methods has been largely shifted to systems of linear differential equations with periodic coefficients (see, for example, [1, 3, 12] and many other works).

Let us dwell in more detail on some of the studies of the Uruguayan mathematician J. L. Massera on the problem of the existence of periodic solutions of ordinary differential periodic systems, which are directly related to this paper. For quite a long time, up to the middle of the 20th century, it was believed in the theory of oscillations that the period of a periodic differential system and the period of its periodic solution are commensurable. And only in 1950 J. L. Massera showed the fallacy of this assumption. Moreover, he obtained (also for linear systems) the conditions for the existence of solutions whose period is incommensurable with the period of the system itself [7]. Subsequently, such solutions, because of their unusual nature, were called *strongly irregular* [2, p. 17].

In the same 1950, J. L. Massera published another paper [8] on the existence of a periodic solution of a periodic differential system of the same period as the system. In particular, he established the following remarkable result: in the linear case the existence of a bounded solution of a periodic system entails the existence of a periodic solution of the same period as the system. In other words, the necessary and sufficient condition for a periodic linear system to have a periodic solution of the same period as the system is the existence of a bounded solution of the system. Consequently, this Massera's theorem reduces the problem of the existence of periodic solution of a periodic linear differential system with the same period as the system to the problem of the existence of a bounded solution. The latter problem is simplier than the original one, since the class of bounded continuously differentiable vector functions is much broader than its subclass consisiting of periodic vector functions.

Thus we have the following rather unexpected property: if a linear periodic system has a solution from a wide class (bounded solutions), it also has a solution from narrow class (periodic solutions of the same period as the system), – a rather rare situation in mathematics in general, if we take into account the fact that only the existence of some object would imply the existence of an object with additional properties. This result of J. L. Massera was transferred or generalized to other types of systems and their solutions in [4–6, 9–11, 13, 14] and others.

As a cosequence of Massera's theorem a natural question arises: is it possible to replace in its formulation the class of bounded solutions by some broader class so that modified theorem remains true. The present paper is devoted to the solution of this problem.

Recall that a set in a topological space is called *nowhere dense* if the interior of its closure is empty, and *a set of the first category according to Baer*, if it can be represented as a countable union of nowhere dense in this space sets. A set that is not a set of the first category is called *a set of the second category according to Baer*.

If M is a topological space and $M_0 \subset M$, then we will say that the space M is *an essential extension* of a subspace M_0 if M_0 has the first category in the space M.

Let M be some set of vector functions defined on the entire numerical axis $\mathbb{R}$. A metric in M given by the equality

$$\operatorname{dist}_{\mathrm{u}}(f,g)=\min\Big\{1,\sup_{t\in\mathbb{R}}\|f(t)-g(t)\|\Big\}\ \text{ for all }\ f,g\in M,$$

is called the metric of *uniform convergence on the axis*, and a metric given by the equality

$$\operatorname{dist}_{\mathrm{c}}(f,g)=\sup_{t\in\mathbb{R}}\min\big\{\|f(t)-g(t)\|,|t|^{-1}\big\}\ \text{ for all }\ f,g\in M,$$

– the metric of *uniform convergence on segments*. It is easy to see that convergence of the sequence $(f_n)_{n\in\mathbb{N}}\subset M$ in the metric $\operatorname{dist}_{\mathrm{u}}$ is equivalent to uniform convergence on the axis, and convergence in the metric $\operatorname{dist}_{\mathrm{c}}$ is equivalent to uniform convergence on each segment.

Next we denote by $\mathcal{B}$ the set of bounded continuously differentiable vector functions $\mathbb{R}\to\mathbb{R}^n$, and by $\mathcal{P}_\omega$ its subset consisting of ω-periodic vector functions. Let us introduce the metric $\operatorname{dist}_{\mathrm{u}}$ of uniform convergence on the axis on the set $\mathcal{B}$ and denote the obtained metric space by $\mathcal{B}_{\mathrm{u}}$.

Consider a linear differential system

$$\dot{x}=A(t)x+f(t),\ \ x\in\mathbb{R}^n,\ \ t\in\mathbb{R}, \tag{1}$$

where $n\in\mathbb{N}$ is fixed, with continuous ω-periodic $n\times n$ coefficient matrix $A(t)$ and free term $f(t)$. Its solutions are continuously differentiable vector functions $x(\,\cdot\,):\mathbb{R}\to\mathbb{R}^n$. As stated above, according to Massera's theorem, if the system (1) has a bounded solution, then it also has an ω-periodic solution. Let us emphasize that we do not assert the ω-periodicity of this bounded solution, but only the fact that the system (1) has an ω-periodic solution. In general, a bounded solution of the ω-periodic system (1) may be neither ω-periodic nor periodic.

The problem stated above has the following formal formulation.

Problem. Is it possible to extend the class $\mathcal{B}$ of bounded vector-functions to some class so that the fact that the ω-periodic system (1) has a solution in this wider class implies that it also has an ω-periodic solution?

Further, while comparing a class of functions and some subclass of it, we will use the language of Baire's categories to understand the relation between them. Thus, Massera's theorem, which reduces the question of the existence of a solution from the set $\mathcal{P}_\omega$ to the question of the existence of a solution from the set $\mathcal{B}$, means that the latter question is much simplier, since, as the following statement shows, the space $\mathcal{B}_{\mathrm{u}}$ is an essential extension of its subspace $\mathcal{P}_\omega$.

Indeed,there is

Proposition. *The set $\mathcal{P}_\omega$ is closed and nowhere dense in the space $\mathcal{B}_{\mathrm{u}}$; in particular, it has the first Baire category in $\mathcal{B}_{\mathrm{u}}$.*

Thus, almost all in the sense of Baire's categories functions of the space $\mathcal{B}_{\mathrm{u}}$ are not ω-periodic. Nevertheless, according to Massera's theorem, only the fact of existing of a solution belonging to the "wide" class (class $\mathcal{B}$) implies the existing of a solution belonging to the "narrow" class (class $\mathcal{P}_\omega$).

Let us give the following

Definition. We will say that a vector function $x(\,\cdot\,):\mathbb{R}\to\mathbb{R}^n$ *grows slower than a linear function*, if at least one of the following relations holds

$$\lim_{t\to-\infty}\frac{\|x(t)\|}{t}=0 \ \text{ or } \ \lim_{t\to+\infty}\frac{\|x(t)\|}{t}=0. \tag{2}$$

We denote by $\mathcal{L}$ the class of continuously differentiable vector functions $\mathbb{R}\to\mathbb{R}^n$, which grow slower than a linear function. Clearly, $\mathcal{B}\subset\mathcal{L}$ and this is a proper inclusion. Indeed, for example, unbounded on $\mathbb{R}$ vector function $(\ln(t^2+1),1,\dots,1)^\top$ satisfies the condition (2), i.e. grows slower than a linear function. Therefore, the following statement strengthens Massera's theorem.

Theorem. *An ω-periodic system* (1) *has an ω-periodic solution if and only if it has a solution that grows slower than a linear function.*

The proof of necessity follows obviously from the chain of inclusions $\mathcal{P}_\omega\subset\mathcal{B}\subset\mathcal{L}$. The proof of sufficiency of the statement of the theorem is equivalent to proving that if the system (1) has no ω-periodic solutions, then it also has no solutions that grow slower than a linear function.

The question naturally arises how significant is extension $\mathcal{L}$ of the set $\mathcal{B}$. If we consider in $\mathcal{L}$ the metric $\mathrm{dist}_{\mathrm{u}}$ of uniform convergence on the axis, then from the point of view of categories there is no difference between $\mathcal{L}$ and $\mathcal{B}$, since, as it is easy to see, in this metric $\mathcal{L}$ is the union of two open sets: the set $\mathcal{B}$ of interest and its complement $\mathcal{L}\setminus\mathcal{B}$.

Consider in $\mathcal{L}$ the metric $\mathrm{dist}_{\mathrm{c}}$ of uniform convergence on segments. We denote the obtained metric space by $\mathcal{L}_{\mathrm{c}}$.

The set $\mathcal{B}$ has the first Baire's category in the space $\mathcal{L}_{\mathrm{c}}$. Thus almost all functions in the metric space $\mathcal{L}_{\mathrm{c}}$ are not bounded on the axis in the sense of categories, i.e. do not belong to the set $\mathcal{B}$. Consequently, the space $\mathcal{L}_{\mathrm{c}}$ is an essential extension of the subspace $\mathcal{B}$.

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