On Solvability Conditions for the Cauchy Problem for Second Order Linear Non-Volterra Functional Differential Equations

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Consider the Cauchy problem for the most general case of linear second order non-Volterra functional differential equations, which can be written in the operator form:

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \ \ t \in [0,1], \\ x(0) = c_0, \quad \dot{x}(0) = c_1, \end{cases} \tag{1}$$

where T^+ and T^- are linear positive operators acting from the space of real continuous functions $\mathbf{C}[0,1]$ into the space of real integrable functions $\mathbf{L}[0,1]$ (positive operators map non-negative functions into non-negative ones), c_0, $c_1 \in \mathbb{R}$, $f \in \mathbf{L}[0,1]$ is integrable.

Let p^+ and p^- be two given non-negative integrable functions. Suppose that positive operators T^+ and T^- satisfy the equalities

$$(T^+\mathbf{1}\,)(t) = p^+(t), \quad (T^-\mathbf{1}\,)(t) = p^-(t), \ \ t \in [0,1], \tag{2}$$

where $\mathbf{1}$ is the unit function, $\mathbf{1}(t) = 1$ for all $t \in [0,1]$. By imposing various restrictions on the functions p^+ and p^-, we can obtain various conditions for the solvability of problem (1) for all operators T^+, T^- satisfying equalities (2) and additional restrictions.

All known solvability conditions of this kind for many boundary value problems were obtained under the same types of restrictions on the operators T^+, T^-, that is only under pointwise restrictions or only under integral ones [2, 4–11]. We can obtain solvability conditions under mixed restrictions, when pointwise restrictions are imposed on the action of one of the operators T^+, T^-, and integral restrictions are imposed on the other operator.

Let us present several obtained statements.

First of all, using ideas of [1, 3, 5, 6], we formulate necessary and sufficient solvability conditions for pointwise restrictions.

Put

$$k(t) \equiv 1 - \int\limits_0^t (t-s)(p^+(s) - p^-(s))\,ds.$$

Theorem 1. *Let non-negative functions p^+, $p^- \in \mathbf{L}[0,1]$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that $T^+\mathbf{1} = p^+$, $T^-\mathbf{1} = p^-$ if and only if*

$$\int\limits_0^1 (1-s)p^+(s)\,ds < 1$$

and

$$\Big(1-\int_0^{t_3}(t_1-s)p^+(s)\,ds+\int_{t_3}^{t_1}(t_1-s)p^-(s)\,ds\Big)k(1)$$
$$+\Big(\int_0^{t_3}(1-s)p^+(s)\,ds-\int_{t_3}^{1}(1-s)p^-(s)\,ds\Big)k(t_1)>0$$

for all $0\le t_3\le t_1\le 1$.

Corollary 1. *Let a non-negative function* $p^-\in\mathbf{L}[0,1]$ *be given.*

The Cauchy problem

$$\begin{cases}\ddot{x}(t)=-(T^-x)(t)+f(t), & t\in[0,1],\\ x(0)=c_0, \quad \dot{x}(0)=c_1,\end{cases}$$

is uniquely solvable for all linear positive operators $T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ *such that* $T^-\mathbf{1}\;=p^-$ *if and only if the inequality*

$$\Delta_-\equiv\Big(1+\int_{t_3}^{t_1}(t_1-s)p^-(s)\,ds\Big)\Big(1+\int_0^{1}(1-s)p^-(s)\,ds\Big)$$
$$-\int_{t_3}^{1}(1-s)p^-(s)\,ds\Big(1+\int_0^{t_1}(t_1-s)p^-(s))\,ds\Big)>0$$

holds for all $0\le t_3\le t_1\le 1$.

Corollary 2. *If*

$$p^-(t)\le 16,\quad p^-(t)\not\equiv 16 \ \ or$$
$$p^-(t)\le 487t^2(1-t)^2 \ \ or \ \ p^-(t)\le 39t \ \ or \ \ p^-(t)\le 24.7e^{-t},$$
$$p^-(t)\le 9.8e^t \ \ or \ \ p^-(t)\le\frac{10.4}{\sqrt{1-t}} \ \ or \ \ p^-(t)\le 32\sin(10\pi t),$$

then the Cauchy problem

$$\begin{cases}\ddot{x}(t)=-(T^-x)(t)+f(t), & t\in[0,1],\\ x(0)=c_0, \quad \dot{x}(0)=c_1\end{cases}$$

is uniquely solvable for all linear positive operators $T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ *such that* $T^-\mathbf{1}\;=p^-$.

With the help of Theorem 1 we can obtain necessary and sufficient solvability conditions for mixed restrictions.

Theorem 2. *Let a non-negative function* $p^-\in\mathbf{L}[0,1]$ *and a number* $\mathcal{P}^+\ge 0$ *be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators* $T^+,T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ *such that*

$$T^-\mathbf{1}\;=p^-,\qquad\int_0^1(1-s)(T^+\mathbf{1}\,)(s)\,ds=\mathcal{P}^+$$

if and only if

$$\mathcal{P}^+ < 1,$$

$$\Delta_-(t_3, t_1, p^-) > \mathcal{P}^+\Big(1 + \int\limits_{t_3}^{t_1} (t_1 - s)p^-(s)\,ds\Big), \ \ 0 \le t_3 \le t_1 \le 1,$$

$$\Delta_-(t_3, t_1, p^-) \ge \mathcal{P}^+\Big(t_1 + (1 - t_1)\int\limits_0^{t_3} sp^-(s)\,ds\Big), \ \ 0 \le t_3 \le t_1 \le 1.$$

Corollary 3. *Let two non-negative numbers $\mathcal{P}^+$, $\mathcal{P}^-$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$\int\limits_0^1 (1 - s)(T^+\mathbf{1})(s)\,ds \le \mathcal{P}^+ \ \ \text{and} \ \ (T^-\mathbf{1})(t) \le \mathcal{P}^-, \ \ t \in [0,1],$$

if and only if

$$\mathcal{P}^+ < 1 \ \ \text{and} \ \ \mathcal{P}^- < 8\Big(1 + \sqrt{1 - \mathcal{P}^+}\Big).$$

Theorem 3. *Let constants $\mathcal{P}^+ \ge 0$, $\mathcal{P}^- \ge 0$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$(T^-\mathbf{1})(t) \le \mathcal{P}^-, \ \ t \in [0,1], \quad \int\limits_0^1 (1 - s)(T^+\mathbf{1})(s)\,ds \le \mathcal{P}^+,$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- < 8\Big(1 + \sqrt{1 - \mathcal{P}^+}\Big).$$

Theorem 4. *Let $\alpha \ge -1$. Let a non-negative function $p^+\mathbf{L}[0,1]$ and a number $\mathcal{P}^- \ge 0$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$T^+\mathbf{1} = p^+, \quad \int\limits_0^1 (1 + \alpha s)(T^-\mathbf{1})(s)\,ds = \mathcal{P}^-$$

if and only if

$$\mathcal{P}^+ \equiv \int\limits_0^1 (1 - s)p^+(s)\,ds < 1,$$

$$\mathcal{P}^- \le \beta(t_1)(1 + t_1\mathcal{T}_3^+ - \mathcal{T}_2^+) + \frac{\mathcal{T}_1^+ - \mathcal{T}_2^+}{t_1} + 2\sqrt{\beta(t_1)(1 - \mathcal{T}_2^+)\Big(\mathcal{T}_3^+ + 1 - \mathcal{T}^+ + \frac{\mathcal{T}_1^+ - \mathcal{T}_2^+}{t_1}\Big)}, \ \ 0 < t_3 \le t_1 < 1,$$

where

$$\beta(t_1) = \frac{1+\alpha t_1}{t_1(1-t_1)},$$

$$\mathcal{T}_1^+ = \int_0^{t_1} (t_1 - s)p^+(s)\,ds, \quad \mathcal{T}_2^+ = \int_0^{t_3} (t_1 - s)p^+(s)\,ds, \quad \mathcal{T}_3^+ = \int_0^{t_3} (1 - s)p^+(s)\,ds.$$

Corollary 4. *Let $\alpha \geq -1$. Let a non-negative function $p^+ \in \mathbf{L}[0,1]$ and a number $\mathcal{P}^- \geq 0$ be given, and $p^+(t) = 0$ for $t \in [0, \frac{1}{1+\sqrt{1+\alpha}}]$.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$T^+\mathbf{1} = p^+, \quad \int_0^1 (1+\alpha s)(T^-\mathbf{1})(s)\,ds = \mathcal{P}^-$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- + 1 - \mathcal{P}^+ \leq \left(1 + \sqrt{1+\alpha} + \sqrt{1-\mathcal{P}^-}\right)^2.$$

Corollary 5. *The Cauchy problem* (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$(T^+\mathbf{1})(t) \leq 2, \quad (T^+\mathbf{1})(t) \not\equiv 2, \quad t \in [0,1],$$

$$\int_0^1 (T^-\mathbf{1})(s)\,ds \leq \min_{t\in(0,1)} \left(\frac{1}{t(1-t)} + t + \sqrt{1+t}\right) \approx 6.9.$$

Corollary 6. *The Cauchy problem* (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$(T^+\mathbf{1})(t) \leq 1, \quad t \in [0,1],$$

$$\int_0^1 (T^-\mathbf{1})(s)\,ds \leq \min_{t\in(0,1)} \left(\frac{1}{t(1-t)} + \frac{t}{2} + \sqrt{\frac{(2-t^2)(1+t)}{t(1-t)}}\right) \approx 7.4.$$

The constants of the solvability conditions from Corollaries 5 and 6 are exact and cannot be increased.

Finally we obtain solvability conditions under integral restrictions on both operators T^+, T^-.

Theorem 5. *Let $\alpha \geq -1$. Let constants $\mathcal{P}^+ \geq 0$, $\mathcal{P}^- \geq 0$ be given.*

The Cauchy Problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$\int_0^1 (1+\alpha s)(T^-\mathbf{1})(s)\,ds \leq \mathcal{P}^-, \quad \int_0^1 (1-s)(T^+\mathbf{1})(s)\,ds \leq \mathcal{P}^+,$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- - \mathcal{P}^+ + 1 \leq \left(1 + \sqrt{1+\alpha} + \sqrt{1-\mathcal{P}^+}\right)^2.$$

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