# Dimensions of Subspaces Defined by the Lyapunov Exponents of Regular Linear Differential Systems with Parametric Perturbations Vanishing at Infinity as Functions of the Parameter

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For a given $n \in \mathbb{N}$ let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

with continuous bounded coefficients defined on the half-axis $\mathbb{R}_+$. In what follows, we identify system (1) with its defining function $A(\,\cdot\,) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and therefore write $A \in \mathcal{M}_n$ and the like. The vector space of solutions to system (1) will be denoted by $\mathcal{S}(A)$. Recall that *the characteristic exponent* (or *the Lyapunov exponent*) of a non-zero solution $x(\,\cdot\,)$ to system (1) is the quantity [7, p. 552], [1, p. 25]

$$\lambda[x] = \varlimsup_{t\to+\infty} \frac{1}{t} \ln \|x(t)\|;$$

for the zero solution let it equal $-\infty$. As is well-known [7, p. 561], [1, p. 38], system (1) has exactly $n$ Lyapunov exponents, counting multiplicity, which we denote by $\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_n(A)$.

For each $\alpha \in \mathbb{R}$, let

$$L_\alpha(A) = \big\{x \in \mathcal{S}(A) : \ \lambda[x] < \alpha\big\} \ \text{ and } \ N_\alpha(A) = \big\{x \in \mathcal{S}(A) : \ \lambda[x] \leqslant \alpha\big\}.$$

Clearly [6, p. 2], for every $\alpha \in \mathbb{R}$, the sets $L_\alpha(A)$ and $N_\alpha(A)$ are vector subspaces of the space $\mathcal{S}(A)$. Let us denote by $d_\alpha(A)$ and $D_\alpha(A)$ respectively their dimensions. In particular, the number $d_0(A)$ is called the *exponential stability index* and, as follows from its definition, coincides with the dimension of the subspace of solutions to system (1) that decay exponentially at infinity.

O. Perron constructed [8], see also [6, p. 13], an example of a two-dimensional diagonal system $A \in \mathcal{M}_2$ and its perturbation $Q \in \mathcal{M}_2$ decaying exponentially at infinity such that the following relations hold:

$$d_a(A) = 1, \quad D_a(A) = 2, \quad d_a(A+Q) = 0, \quad D_a(A+Q) = 1, \tag{2}$$

where $a$ is a positive number. Moreover, it is fairly easy to see that in equalities (2) the number $a$ can be taken arbitrary. This assertion follows from an obvious fact that by adding the matrix $\gamma I_n$ ($I_n$ being the $n \times n$ identity matrix and $\gamma \in \mathbb{R}$) to the coefficient matrix $A$ of system (1), we change the Lyapunov exponents of all its solutions by $\gamma$.

Thus, by virtue of the Perron example, the quantities $d_\alpha$ and $D_\alpha$ are not invariant under vanishing at infinity perturbations of system coefficients and hence [9, Lemma 7.3], they are not semicontinuous in the topology of uniform convergence over the half-axis $\mathbb{R}_+$ on the space $\mathcal{M}_n$.

System (1) is said [7, p. 563], [1, p. 61] to be *regular*, if the following two conditions are met:

1) the limit
$$T(A) = \lim_{t\to+\infty} t^{-1} \int\limits_0^t \operatorname{tr} A(\tau)\, d\tau$$
exists, where $\operatorname{tr}(\,\cdot\,)$ stands for the trace of a matrix;

2) the equality
$$\lambda_1(A) + \lambda_2(A) + \cdots + \lambda_n(A) = T(A)$$
holds.

The class of regular $n$-dimensional systems will be denoted by $\mathcal{R}_n$.

A. M. Lyapunov demonstrated [7, pp. 576–578] that if a nonlinear system (under natrural assumptions on the right-hand side) has a regular first approximation system with the exponential stability index equal to $k \in \{1, \dots, n\}$, then the nonlinear system possesses exactly $k$-dimensional exponentially stable manifold passing through the origin (i.e. any solution to the nonlinear system starting on this manifold decays exponentially; furthermore, such a solution $x(\,\cdot\,)$ admits the estimate
$$\|x(t)\| \leqslant C_\varepsilon \exp\bigl\{(\lambda_k(A) + \varepsilon)t\bigr\}\|x(0)\| \ \text{ for every } \ \varepsilon > 0,$$
where $\lambda_k(A)$ is the $k$-th Lyapunov exponent of the first approximation system). Taking into account this fundamental result, one may conjecture that the exponential stability index of a regular system (and along with it the quantities $d_\alpha$ and $D_\alpha$ for all $\alpha \in \mathbb{R}$) is invariant under vanishing at infinity perturbations of its coefficients. Let us note that for exponentially decaying perturbations of a regular system the mentioned invariance does indeed take place [3, 4].

The conjecture stated above had been around for quite some time, until R. È. Vinograd in the paper [10] gave an example of systems $A, B \in \mathcal{R}_2$ for which the relations
$$d_0(A) = 0, \quad D_0(A) = 2, \quad D_0(B) = d_0(B) = 1, \quad \lim_{t\to+\infty} \|A(t) - B(t)\| = 0$$
are valid.

Let $M$ be a metric space. Consider a family of systems
$$\dot{x} = \mathcal{A}(t, \mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \tag{3}$$
such that for each fixed $\mu \in M$ the matrix-valued function $\mathcal{A}(\,\cdot\,, \mu)$ has continuous and bounded coefficients, i.e. $\mathcal{A}(\,\cdot\,, \mu) \in \mathcal{M}_n$.

Here and subsequently, $\mathcal{Z}_n$ stands for the set $\{0, 1, \dots, n\}$. For each $\alpha \in \mathbb{R}$, define the functions $d_\alpha(\,\cdot\,; \mathcal{A}), D_\alpha(\,\cdot\,; \mathcal{A}) : M \to \mathcal{Z}_n$ by
$$d_\alpha(\mu; \mathcal{A}) = d_\alpha(\mathcal{A}(\,\cdot\,, \mu)) \ \text{ and } \ D_\alpha(\mu; A) = D_\alpha(\mathcal{A}(\,\cdot\,, \mu)), \ \ \mu \in M.$$

Let $\mathcal{R}^n(M)$ denote the class of families of systems (1) with coefficient matrices of the form $\mathcal{A}(t, \mu) = B(t) + Q(t, \mu)$, where a matrix-valued function $B : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded, the system $\dot{x} = B(t)x$ is regular, and $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is continuous and satisfies the condition
$$\sup_{\mu\in M} \|Q(t, \mu)\| \to 0 \ \text{ as } \ t \to +\infty.$$

Given $\alpha, \beta \in \mathbb{R}$, let

$$R^n_{\alpha,\beta}(M) \equiv \big\{(d_\alpha(\,\cdot\,,\mathcal{A}), D_\beta(\,\cdot\,,\mathcal{A})):\ \mathcal{A} \in \mathcal{R}^n(M)\big\}.$$

The problem is to obtain a complete function-theoretic description of the classes $R^n_{\alpha,\beta}(M)$ for any metric space $M$ and numbers $n \geq 2$ and $\alpha, \beta \in \mathbb{R}$. This problem can be viewed as a generalization of Vinograd's example [10] of instability of the Lyapunov exponents of a regular system under vanishing at infinity perturbations of its coefficient matrix.

Following [5, p. 264], for a number $r \in \mathbb{R}$ and function $f : M \to \mathbb{R}$, we write $[f \geqslant r]$ for the *Lebesgue* set $\{\mu \in M :\ f(\mu) \geq r\}$.

Before stating the main result of the report, let us recall [5, p. 156] that a subset of a metric space is said to be an $F_\sigma$-set, if it can be represented as a countable union of closed subsets, and an $F_{\sigma\delta}$-set, if it can be represented as a countable intersection of $F_\sigma$-sets.

The following statement solves the problem posed above.

**Theorem.** *For any metric space $M$, real numbers $\alpha$, $\beta$ and integer $n \geq 2$, a vector function $(g,h) : M \to \mathcal{Z}_n \times \mathcal{Z}_n$ belongs to the class $R^n_{\alpha,\beta}(M)$, if and only if for every $r \in \mathbb{R}$, the set $[g \geq r]$ is an $F_\sigma$-set and $[h \geq r]$ is an $F_{\sigma\delta}$-set, and for all $\mu \in M$, we have either $h(\mu) \geq g(\mu)$ or $h(\mu) \leq g(\mu)$, depending on whether $\beta \geq \alpha$ or $\beta < \alpha$.*

**Remark 1.** A complete description of an analogous class of vector functions corresponding to families (3) with coefficients of the form $\mathcal{A}(t,\mu) = B(t) + Q(t,\mu)$, where $B : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded, and $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is continuous and decays exponentially (uniformly in $\mu$) as $t \to +\infty$, is obtained in the paper [2] and coincides with the one stated above.

**Remark 2.** The class $R^1_{\alpha,\beta}(M)$ consists of pairs of constant functions $M \to \{0,1\}$, namely:

$$R^1_{\alpha,\beta}(M) = \begin{cases} \{(0,0),(1,1),(0,1)\}, & \text{if } \beta \geq \alpha, \\ \{(0,0),(1,1),(1,0)\}, & \text{if } \beta < \alpha. \end{cases}$$

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