On the Criterion of Well-Posedness of the Modified Cauchy Problem for Singular Systems of Linear Ordinary Differential Equations

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Let $[a,b]\subset\mathbb{R}$ be a finite and closed interval non-degenerated in the point.

Consider the modified initial problem for a linear system of generalized ordinary differential equations with singularities

$$dx=dA(t)\cdot x+df(t)\ \text{ for }\ t\in[a,b[\,, \tag{1}$$

$$\lim_{t\to b-}\big(\Phi^{-1}(t)\,x(t)\big)=0, \tag{2}$$

where $A=(a_{ik})_{i,k=1}^n$ is an $n\times n$-matrix valued function and $f=(f_k)_{k=1}^n$ is an n-vector valued function, both of them have a locally bounded variation on $[a,b[$; $\Phi=\operatorname{diag}(\varphi_1,\dots,\varphi_n)$ is a diagonal $n\times n$-matrix valued function, defined on $[a,b[$ and having an inverse $\Phi^{-1}(t)$ for each $t\in[a,b[\,$.

Along with system (1) consider the perturbed singular systems

$$dx=dA_m(t)\cdot x+df_m(t)\ \text{ for }\ t\in[a,b[\tag{3}$$

$(m=1,2,\dots)$ under conditions (2), where A_m is an $n\times n$-matrix valued function and f_m is an n-vector valued function, both of them have a locally bounded variation on $[a,b[\,$.

We are interested to established the necessary and sufficient conditions whether the unique solvability of problem $(1),(2)$ guarantees the unique solvability of problem $(3),(2)$ and nearness of its solution in the definite sense if matrix-functions A_m and A and vector-functions f_m and f are nearly among themselves.

We assume $A(a)=A_m(a)=O_{n\times n}$ and $f(a)=f_m(a)=0_n$ $(m=1,2,\dots)$ without loss of generality.

The same and related problems for ordinary differential systems with singularities $\frac{dx}{dt}=P(t)\,x+q(t)$, where $P\in L_{loc}([a,b[,\mathbb{R}^{n\times n})$, $q\in L_{loc}([a,b[,\mathbb{R}^n)$, have been investigated in [7,9] (see, also, the references therein).

The singularity of system (1) consists in the fact that both A and f need not to have bounded variations on any interval containing the point t_0.

The solvability question of the generalized differential problem $(1),(2)$ has been investigated in [6]. The well-posedness of problem $(1),(2)$ with singularity has been considered in [4]. To our knowledge, the necessary and sufficient conditions for well-posedness of problem $(1),(2)$ with singularity has not been investigated up to now.

Some singular boundary problems for the generalized differential system (1) are investigated in [1,2] (see, also, the references therein).

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1–6, 8, 10, 11] and the references therein).

In the paper, we give necessary and sufficient conditions for the so called strongly Φ-well-posedness of problem (1), (2).

Throughout the paper we use the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$. $\mathbb{R}_+ =]0, +\infty[$. $\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices with the standard norm.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all column n-vectors $x = (x_i)_{i=1}^n$.

If $X = (x_{ik})_{i,k=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ik}|)_{i,k=1}^{n,m}$, $[X]_\mp = \frac{1}{2}\,(|X| \mp X)$.

$O_{n\times m}$ (or O) is the zero $n \times m$-matrix, 0_n (or 0) is the zero n-vector.

I_n is identity $n \times n$-matrix.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X.

The inequalities between the matrices are understood componentwisely.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X : \mathbb{R} \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a, b]$ of its components; $\bigvee\limits_a^{b-}(X) = \lim\limits_{t\to b-} \bigvee\limits_a^t(X)$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$ at the point t.

$\mathrm{BV}([c, d], \mathbb{R}^{n\times m})$ is the set of bounded variation matrix-functions on $[c, d]$.

$\mathrm{BV}_{loc}([a, b[; \mathbb{R}^{n\times m})$ is the set of all locally bounded matrix-functions.

If $X \in \mathrm{BV}_{loc}([a, b[; \mathbb{R}^{n\times m})$, then

$$[X(t)]_-^v \equiv \frac{1}{2}\,(V(X)(t) - X(t)), \quad [X(t)]_+^v \equiv \frac{1}{2}\,(V(X)(t) + X(t)).$$

$s_1, s_2, s_c : \mathrm{BV}_{loc}([a, b[; \mathbb{R}) \to \mathrm{BV}_{loc}([a, b[; \mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0, \quad s_c(x)(a) = x(a),$$
$$s_1(x)(t) = s_1(x)(a) + \sum_{a<\tau\le t} d_1x(\tau), \quad s_2(x)(t) = s_2(x)(a) + \sum_{a\le\tau<t} d_2x(\tau),$$
$$s_c(x)(t) = s_c(x)(a) + x(t) - x(a) - \sum_{j=1}^{2} s_j(x)(t) \ \text{ for } \ a < t < b.$$

If $g : [a, b] \to \mathbb{R}$ is a nondecreasing function and $x : [a, b] \to \mathbb{R}$, then

$$\int\limits_s^t x(\tau)\,dg(\tau) = \int\limits_{]s,t[} x(\tau)\,ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)\,d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)\,d_2g(\tau)$$
$$\text{for } \ s < t;\ s, t \in [a, b],$$

where $\int\limits_{]s,t[} x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_c(g)$. So $\int\limits_s^t x(\tau)\,dg(\tau)$ is the Kurzweil integral ([10,11]).

We put $\int\limits_s^{t-} x(\tau)\,dg(\tau) = \lim_{\delta\to 0+} \int\limits_s^{t-\delta} x(\tau)\,dg(\tau)$.

If $G=(g_{ik})_{i,k=1}^{l,n} : [a,b]\to\mathbb{R}^{l\times n}$ and $X=(x_{kj})_{k,j=1}^{n,m} : [a,b]\to\mathbb{R}^{n\times m}$, then

$$\int\limits_a^t dG(\tau)\cdot X(\tau) \equiv \bigg(\sum_{k=1}^n \int\limits_a^t x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

We introduce the operators $\mathcal{A}(X,Y)$, $\mathcal{B}(X,Y)$ and $\mathcal{I}(X,Y)$ in the following way:

(a) if $X\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^j d_jX(t))\neq 0$ for $t\in I$ $(j=1,2)$, and $Y\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, then $\mathcal{A}(X,Y)(a)=O_{n\times m}$,

$$\mathcal{A}(X,Y)(t)\equiv Y(t)-Y(a)+\sum_{a<\tau\leq t} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)$$
$$-\sum_{a\leq\tau<t} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau);$$

(b) if $X\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Y:I\to\mathbb{R}^{n\times m}$, then $\mathcal{B}(X,Y)(a)=O_{n\times m}$,

$$\mathcal{B}(X,Y)(t)\equiv X(t)Y(t)-X(a)Y(a)-\int\limits_a^t dX(\tau)\cdot Y(\tau);$$

(c) if $X\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(X(t))\neq 0$, and $Y:I\to\mathbb{R}^{n\times n}$, then

$$\mathcal{I}(X,Y)(a)=O_{n\times m},\ \ \mathcal{I}(X,Y)(t)\equiv \int\limits_a^t d\big(X(\tau)+\mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau).$$

In addition, let $\mathcal{V}_j(\Phi,A_*,\cdot\,):\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times l})\to\mathbb{R}$ $(j=1,2)$ be operators defined, respectively, by

$$\mathcal{V}_1(\Phi,A_*,F)(t,\tau)=\int\limits_t^\tau \Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*,F))(s)\cdot\Phi(s)\ \text{ and}$$

$$\mathcal{V}_2(\Phi,A_*,F)(t,\tau)=\int\limits_t^\tau \Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*,A_*))(s)\cdot|F(s)|\ \text{ for }\ a\leq t<\tau<b.$$

A vector-function $x:I\to\mathbb{R}^n$ is said to be a solution of system (1) if $x\in\mathrm{BV}_{loc}(I,\mathbb{R}^n)$ and

$$x(t)=x(a)+\int\limits_a^t dA(\tau)\cdot x(\tau)+f(t)-f(a)\ \text{ for }\ t\in I.$$

We assume that $\det(I_n + (-1)^j d_j A(t)) \neq 0$ for $t \in I$ $(j = 1, 2)$.

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e., for the case when $A \in \mathrm{BV}([a, c]; \mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}([a, c]; \mathbb{R}^n)$ for every $c \in I$.

Let a matrix-function $A_* = (a_{*ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ be such that $\det(I_n+(-1)^j d_j A_*(t)) \neq 0$ for $t \in I$ $(j = 1, 2)$.

Then a matrix-function $C_* : I \times I \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system $dx = dA_*(t) \cdot x$, if, for each interval $J \subset I$ and $\tau \in J$, the restriction of the matrix-function $C_*(\,\cdot\,, \tau) : I \to \mathbb{R}^{n\times n}$ on J is the fundamental matrix of the system, satisfying the condition $C_*(\tau, \tau) = I_n$. Therefore, C_* is the Cauchy matrix of the system if and only if the restriction of C_* on $J \times J$ is the Cauchy matrix of the system in the regular case. Let $X_*(\tau) \equiv C_*(\,\cdot\,, \tau)$.

Definition 1. Problem (1), (2) is said to be weakly Φ-well-posed with respect to the matrix-function A_* if it has the unique solution x_0 and for every sequences of A_m and f_m $(m = 1, 2, \dots)$ such that

$$\det\big(I_n + (-1)^j d_j A_m(t)\big) \neq 0 \ \text{ for } \ t \in I \ \ (j = 1, 2), \tag{4}$$

for each sufficiently large m, and the conditions

$$\lim_{m\to+\infty} \big\|\mathcal{V}_1(\Phi, A_*, A_m - A)(t, b-)\big\| = 0, \tag{5}$$

$$\lim_{m\to+\infty} \big\|\mathcal{V}_2(\Phi, A_*, f_m - f)(t, b-)\big\| = 0, \tag{6}$$

$$\lim_{m\to+\infty} \big\|\Phi^{-1}(t)(f_m(t) - f(t)) - \Phi^{-1}(b-)(f_m(b-) - f(b-))\big\| = 0 \tag{7}$$

hold uniformly on I, problem (3), (2) has the unique solution x_m for each sufficiently large m and

$$\lim_{m\to+\infty} \big\|\Phi^{-1}(t)\,(x_m(t) - x_0(t))\big\| = 0 \ \text{ uniformly on } I. \tag{8}$$

Definition 2. Problem (1), (2) is said to be strongly Φ-well-posed with respect to the matrix-function A_* if it has the unique solution x_0 and for every sequences of matrix-and vector-functions A_m and f_m $(m = 1, 2, \dots)$ such that condition (4) holds for every sufficiently large m and the conditions (6) and

$$\lim_{m\to+\infty} \big\|\mathcal{V}_1(\Phi, A_*, f_m - f)(t, b-)\big\| = 0$$

hold uniformly on I, problem (3), (2) has the unique solution x_m for each sufficiently large m and condition (8) holds.

Remark 1. If problem (1), (2) is strongly well-posed, then it is weakly well-posed, as well, because

$$\begin{aligned}\big\|\mathcal{V}_1(\Phi, A_*, f_m - f)(t, \tau)\big\| \le \big\|\Phi^{-1}(t)(f_m(t) - f(t)) - \Phi^{-1}(\tau)(f_m(\tau) - f(\tau))\big\| \\ + \big\|\mathcal{V}_2(\Phi, A_*, f_m - f)(t, \tau)\big\| \ \text{ for } \ a \le t < \tau < b.\end{aligned}$$

Definition 3. We say that the sequence (A_m, f_m) $(m = 1, 2, \dots)$ belongs to the set $\mathcal{S}_{A_*}(A, f; \Phi, b)$, i.e.,

$$\big((A_m, f_m)\big)_{m=1}^{+\infty} \in \mathcal{S}_{A_*}(A, f; \Phi), \tag{9}$$

if problem (3), (2) has the unique solution x_m for each sufficiently large m and condition (8) holds.

Let $I(\delta) = [b - \delta, b[$ for every $\delta > 0$.

Theorem 1. *Let there exist nonnegative constant $n\times n$ matrices B_0 and B such that*

$$r(B)<1, \tag{10}$$

the estimates $|C_*(t,\tau)|\le\Phi(t)B_0\Phi^{-1}(\tau)$ for $b-\delta\le t\le\tau<b$ and

$$\bigg|\int\limits_t^{b-}|C_*(t,s)|\,d\,\mathrm{V}(\mathcal{A}(A_*,A-A_*))(s)\cdot\Phi(s)\bigg|\le H(t)B\ \text{ for }\ t\in I(\delta)$$

fulfilled for some $\delta>0$. Let, moreover,

$$\lim_{t\to b-}\bigg\|\int\limits_t^{b-}\Phi^{-1}(t)\,C_*(t,\tau)\,d\mathcal{A}(A_*,f)(\tau)\bigg\|=0.$$

Then problem (1), (2) *is weakly Φ-well-posed with respect to A_*.*

Theorem 2. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ such that conditions* (10) *and*

$$\big[(-1)^j d_j a_{ii}(t)\big]_+>-1\ \text{ for }\ t\in I\ \ (j=1,2;\ i=1,\dots,n)$$

hold, and the estimates

$$c_i(t,\tau)\le b_0\,\frac{h_i(t)}{h_i(\tau)}\ \text{ for }\ b-\delta\le t\le\tau<b\ \ (i=1,\dots,n);$$

$$\bigg|\int\limits_t^{b-}c_i(t,\tau)h_i(\tau)\,d[a_{ii}(\tau)]_-^v\bigg|\le b_{ii}\,h_i(t)\ \text{ for }\ t\in I(\delta)\ \ (i=1,\dots,n),$$

$$\bigg|\int\limits_t^{b-}c_i(t,\tau)h_k(\tau)\,d\,\mathrm{V}(\mathcal{A}(a_{*ii},a_{ik}))(\tau)\bigg|\le b_{ik}h_i(t)\ \text{ for }\ t\in I(\delta)\ \ (i\ne k;\ i,k=1,\dots,n)$$

fulfilled for some $b_0>0$ and $\delta>0$. Let, moreover,

$$\lim_{t\to b-}\int\limits_t^{b-}\frac{c_i(t,\tau)}{h_i(t)}\,d\,\mathrm{V}(\mathcal{A}(a_{*ii},f_i))(\tau)=0\ \ (i=1,\dots,n),$$

where $a_{*ii}(t)\equiv[a_{ii}(t)]_+^v$ $(i=1,\dots,n)$, and c_i is the Cauchy function of the equation $dx=x\,da_{*ii}(t)$. Then problem (1), (2) is weakly Φ-well-posed with respect to the matrix-function $A_*(t)\equiv$ $\operatorname{diag}(a_{*11}(t),\dots,a_{*nn}(t))$.

Theorem 3. *Let conditions of Theorem 1 be fulfilled and let there exist a sequence of non-degenerated matrix-functions $H_m\in\mathrm{BV}_{loc}([a,b[;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that*

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t)-I_n\big\|=0, \tag{11}$$

$$\lim_{m\to+\infty}\big\|\mathcal{V}_1(\Phi,A_*,A_m^*-A)(t,b-)\big\|=0, \tag{12}$$

$$\lim_{m\to+\infty}\big\|\mathcal{V}_2(\Phi,A_*,f_m^*-f)(t,b-)\big\|=0, \tag{13}$$

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)(f_m^*(t)-f(t))-\Phi^{-1}(b-)(f_m^*(b-)-f(b-))\big\|=0 \tag{14}$$

hold uniformly on I, where $A_m^*(t)\equiv\mathcal{I}(H_m,A_m)(t)$ and $f_m^*(t)\equiv\mathcal{B}(H_m,f_m)(t)$. Then inclusion $((A_m^*,f_m^*))_{m=1}^{+\infty}\in\mathcal{S}_{A_*}(A,f;\Phi)$ holds.

Theorem 3 has the following form for $H_m(t) \equiv I_n$ $(m = 1, 2, \dots)$.

Corollary 1. *Let conditions of Theorem 1 be fulfilled and conditions* (5)–(7) *hold uniformly on* I*. Then inclusion* (9) *holds.*

Theorem 4. *Let conditions of Theorem 1 be fulfilled and let, moreover,*

$$\|B_0\| \, \|(I_n - B)^{-1}\| < 1 \tag{15}$$

and

$$\limsup_{t\to b-} \bigg\| \Phi^{-1}(t) \int\limits_t^{b-} dV(A)(s)\cdot\Phi(s) \bigg\| < +\infty.$$

Then inclusion (9) *holds if and only if there exist the sequence of matrix functions* $H_m \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(m = 1, 2, \dots)$ *such that*

$$\limsup_{t\to b-} \bigg\| \int\limits_t^{b-} \Phi^{-1}(s)\, d\,\mathrm{V}(\mathcal{A}(A_*, A_*))(s)\cdot\Phi(s) \bigg\| < +\infty \ \text{ for } \ a \le t < \tau < b,$$

$$\limsup_{t\to b-} \bigg(\|\Phi^{-1}(t)(f^*_m(t) - f(t))\| + \bigg\| \Phi^{-1}(t) \int\limits_t^{b-} d\,\mathrm{V}(A)(s)\cdot |f^*_m(s) - f(s)| \bigg\| \bigg) = 0 \tag{16}$$

and conditions (11)–(14) *hold uniformly on* I*, where the matrix- and vector functions* A^*_m *and* f^*_m $(m = 1, 2, \dots)$ *are defined as in Theorem 3.*

Theorem 4′. *Let conditions of Theorem 4 be fulfilled. Then inclusion* (9) *holds if and only if conditions* (13), (14) *and*

$$\lim_{m\to+\infty} \big\| \Phi^{-1}(t)(X_m(t) - X_0(t)) \big\| = 0$$

hold uniformly on I*, where* X_0*,* X_m *are the fundamental matrices of systems* (1), (3)*, respectively, and* $f^*_m(t) \equiv \mathcal{B}(X_0 X_m^{-1}, f_m)(t)$ $(m = 1, 2, \dots)$.

Remark 2. In Theorem 4, condition (15) is essential and it cannot be neglected, i.e., if the condition is violated, then the conclusion of the theorem is not true, in general. Below we present an example.

Let $I = [0, 1]$, $n = 1$, $b = 1$, $B = 0$, $B_0 = 1$, $\Phi(t) \equiv 1 - t$; $A(t) = A_m(t) = A_*(t) \equiv \ln(1 - t)$ $(m = 1, 2, \dots)$;

$$f(t) \equiv 0, \quad f_m(t) \equiv -\frac{1}{m} \int\limits_0^t \cos \frac{\ln(1-t)}{m} \quad (m = 1, 2, \dots).$$

Then $C_*(t,\tau) \equiv 1 - t(1-\tau)^{-1}$, $x_0(t) \equiv 0$, $x_m(t) \equiv (1-t)\sin\frac{\ln(1-t)}{m}$ $(m = 1, 2, \dots)$. So, all conditions of Theorem 4 are fulfilled, except of (15), but condition (8) is not fulfilled uniformly on I.

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