Exact Baire Class of the Local Entropy Considered as a Function of a Point in the Phase Space

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Following [1], we give the definition of the local entropy that will be necessary in what follows. Let X be a compact metric space with a metric d and $f : X \to X$ a continuous map. Along with the original metric d, we define an additional system of metrics on X:

$$d_n^f(x,y) = \max_{0\le i\le n-1} d(f^i(x), f^i(y)), \quad x, y \in X, \quad n \in \mathbb{N},$$

where f^i, $i \in \mathbb{N}$, is the i-th iteration of f, $f^0 \equiv \mathrm{id}_X$. Given a point $x \in X$, for any $n \in \mathbb{N}$, $r > 0$ and $\rho > 0$, denote by $N_d(f, r, n, x, \rho)$ the maximum number of points in the ball $B_d(x,\rho) = \{y \in X : \ d(x,y) < \rho\}$, pairwise d_n^f-distances between which are greater than r. Then *the local entropy* of the mapping f at the point x is defined by the formula

$$h_d(f,x) = \lim_{r\to 0}\lim_{\rho\to 0}\varlimsup_{n\to\infty}\frac{1}{n}\ln N_d(f,r,n,x,\rho).$$

Recall one more formula for calculating the local entropy. For any r, $\rho > 0$ and $n \in \mathbb{N}$ a set $A \subset B_d(x,\rho)$ is called an (f,r,n,x,ρ)-cover of the ball $B_d(x,\rho)$, if for any point $y \in B_d(x,\rho)$ there is a point $z \in A$ such that $d_n^f(z,y) < r$. Let $S_d(f,r,n,x,\rho)$ denote the minimum number of elements in an (f,r,n,x,ρ)-cover, then the local entropy can be calculated by the formula

$$h_d(f,x) = \lim_{r\to 0}\lim_{\rho\to 0}\varlimsup_{n\to\infty}\frac{1}{n}\ln S_d(f,r,n,x,\rho). \tag{1}$$

For a fixed continuous mapping $f : X \to X$, consider the function

$$x \mapsto h_d(f,x). \tag{2}$$

As the following example shows, function (2) can be discontinuous on the space X. Let $X = [-1,1]$ and define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in [-1,0), \\ 4x(1-x), & \text{if } x \in [0,1]. \end{cases}$$

Then $h_d(f,x) = 0$ for $x \in [-1,0)$ and $h_d(f,0) = \ln 2$, hence function (2) has a discontinuity at zero.

Recall that continuous functions on a metric space $\mathcal{M}$ are called functions of the zeroth Baire class, and for every natural number p, functions of the p-th Baire class are those that are pointwise limits of sequences of functions in the $(p-1)$-th class.

There are many, not equivalent to each other, interpretations as to which properties are typical and which are not. Here we recall the notion of typicality introduced and studied by R.-L. Baire. A property of a point in a topological space is called Baire typical if the set of points possessing this property contains an everywhere dense G_δ-set.

Theorem 1 ([2]). *For any continuous mapping $f : X \to X$, function (2) belongs to the second Baire class and is lower semicontinuous at a Baire typical point of the space X.*

Proof. Let us transform formula (1) to the form

$$h_d(f,x) = \lim_{m\to\infty} \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S_d\Big(f, \frac{1}{m}, n, x, \frac{1}{k}\Big), \tag{3}$$

and for a fixed natural number m consider the function

$$x \mapsto \varphi_m(x) = \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S_d\Big(f, \frac{1}{m}, n, x, \frac{1}{k}\Big).$$

For any $k > 0$ and any point $y \in B_d(x, 1/k)$, there exists $l_k > 0$ such that for all $l \geqslant l_k$ the inclusion

$$B_d\Big(y, \frac{1}{l}\Big) \subset B_d\Big(x, \frac{1}{k}\Big)$$

holds, which implies the inequality

$$S_d\Big(f, \frac{1}{m}, n, y, \frac{1}{l}\Big) \leqslant S_d\Big(f, \frac{1}{m}, n, x, \frac{1}{k}\Big), \quad m, n \in \mathbb{N}.$$

Consequently,

$$\lim_{l\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S_d\Big(f, \frac{1}{m}, n, y, \frac{1}{l}\Big) \leqslant \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S_d\Big(f, \frac{1}{m}, n, x, \frac{1}{k}\Big).$$

Since the point $y \in B_d(x, 1/k)$ is arbitrary, we obtain the inequality

$$\sup_{y\in B_d(x,1/k)} \lim_{l\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S_d\Big(f, \frac{1}{m}, n, y, \frac{1}{l}\Big) \leqslant \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S_d\Big(f, \frac{1}{m}, n, x, \frac{1}{l}\Big).$$

Passing in the last inequality to the limit as $k \to +\infty$, we obtain the inequality

$$\overline{\lim_{y\to x}} \varphi_m(x) \leqslant \varphi_m(x),$$

which establishes upper semicontinuity of the function $x \mapsto \varphi_m(x)$ at the point x. Hence the function $x \mapsto \varphi_m(x)$ belongs to the first Baire class on the space X. Thus, from (3) we get the following representation of the local entropy of the continuous mapping f at the point x:

$$h_d(f,x) = \lim_{m\to\infty} \varphi_m(x), \quad \varphi_1(x) \leqslant \varphi_2(x) \leqslant \varphi_3(x) \leqslant \cdots,$$

which implies that the function $x \mapsto h_d(f,x)$ belongs to the second Baire class on the space X.

By the Baire theorem on functions of the first class, for each $m \in \mathbb{N}$, the set of points of continuity G_m for the function $x \mapsto \varphi_m(x)$ is an everywhere dense G_δ-set. The intersection of all G_m is again an everywhere dense set, each point of which is a point of continuity for all functions $x \mapsto \varphi_m(x)$, $m \in \mathbb{N}$. Let $x \in \bigcap\limits_{m\in\mathbb{N}} G_m$ and $\varepsilon > 0$. By definition of the limit, $\varphi_m(x) \geqslant h_d(f,x) - \varepsilon$ for all sufficiently large m. Fixing such m, find a neighborhood $B_d(x,\delta)$ of the point x such that for every $y \in B_d(x,\delta)$ we have $\varphi_m(y) \geqslant \varphi_m(x) - \varepsilon$. Since the sequence (φ_m) is nondecreasing, it follows that $h_d(f,y) \geqslant \varphi_m(y)$ for all $y \in B_d(x,\delta)$, hence $\varphi_m(y) \geqslant h_d(f,x) - 2\varepsilon$. Therefore, at each point of the set $\bigcap\limits_{m\in\mathbb{N}} G_m$ the function $x \mapsto h_d(f,x)$ is lower semicontinuous. $\square$

On the set of sequences $x=(x_1,x_2,\dots)$, $x_k\in\{0,1\}$, introduce a metric

$$d_{\Omega_2}(x,y)=\begin{cases}0, & \text{if } x=y,\\ \dfrac{1}{\min\{i:\ x_i\neq y_i\}}, & \text{if } x\neq y.\end{cases}$$

The resulting compact metric space will be denoted by Ω_2. Note that the space Ω_2 is homeomorphic to the Cantor set on the segment $[0,1]$ with the metric induced by the natural metric of the real line.

Theorem 2 ([2])**.** *If $X=\Omega_2\times\Omega_2$ with the metric*

$$d\big((x,\alpha),(y,\beta)\big)=\max\big\{d_{\Omega_2}(x,y),d_{\Omega_2}(\alpha,\beta)\big\},$$

then there is a continuous mapping $f:X\to X$ such that function (2) *is everywhere discontinuous and does not belong to the first Baire class on the space X.*

Proof. Define a mapping $f:\Omega_2\times\Omega_2\to\Omega_2\times\Omega_2$ as follows:

$$f\big((x_1,x_2,x_3,\dots),(\alpha_1,\alpha_2,\alpha_3,\dots)\big)=\Big((x_{1+\alpha_1},x_{2+\alpha_2},x_{3+\alpha_3},\dots),(\alpha_1,\alpha_2,\alpha_3,\dots)\Big).$$

Denote by $\mathcal{P}_0$ the set of sequences from Ω_2 for which all but a finite number of terms are equal to zero, and by $\mathcal{P}_1$ the set of sequences from Ω_2 for which all but a finite number of terms are equal to one.

Lemma 1. *For any point $(x,\alpha)\in\Omega_2\times P_0$, the equality $h_d(f,(x,\alpha))=0$ is valid.*

Proof. If $(\alpha_1,\alpha_2,\alpha_3,\dots)\in P_0$, then there is a natural number p_0 such that $\alpha_p=0$ for all $p\geqslant p_0$. Therefore, for any $m\geqslant p_0$ and $(y,\beta)\in B_d((x,\alpha),\frac{1}{m+1})$,

$$\begin{aligned}&f(y,\beta)\\&\quad=\Big(\Big(x_{1+\alpha_1},\dots,x_{p_0+\alpha_{p_0}},x_{p_0},\dots,x_m,y_{m+1+\beta_{m+1}},y_{m+2+\beta_{m+2}},\dots\Big),(\alpha_1,\dots,\alpha_m,\beta_{m+1},\dots)\Big),\end{aligned}$$

therefore d_n^f-distance between any two points of the ball $B_d((x,\alpha),\frac{1}{m+1})$ does not exceed $\frac{1}{m+1}$. Thus, for any $k>m$ we have

$$N_d\Big(f,\frac{1}{m},n,(x,\alpha),\frac{1}{k}\Big)=1,$$

and hence

$$h_d(f,(x,\alpha))=0. \qquad\square$$

Lemma 2. *For any point $(x,\alpha)\in\Omega_2\times P_1$, the inequality $h_d(f,(x,\alpha))\geqslant\ln 2$ is valid.*

Proof. If $(\alpha_1,\alpha_2,\alpha_3,\dots)\in P_1$, then there is a natural number p_0 such that $\alpha_p=1$ for all $p\geqslant p_0$ and hence for any point $(x,\alpha)\in\Omega_2\times P_1$ we have the equality

$$f(x,\alpha)=\big((x_{1+\alpha_1},\dots,x_{p_0-1+\alpha_{p_0-1}},x_{p_0+1},x_{p_0+2},\dots),\alpha\big).$$

In the ball $B_d((x,\alpha),\frac{1}{p})$ for each natural number $n\geqslant p+2$, consider the set $A_{n,p}$ of points of the form

$$\big((x_1,\dots,x_p,y_{p+1},\dots,y_n,0,0,\dots),\alpha\big),\quad\text{where}\quad y_i\in\{0,1\},\quad i=p+1,\dots,n.$$

Since the d_n^f-distance between any two points from $A_{n,p}$ is not less than $\frac{1}{p+1}$, then the quantity $N_d(f, \frac{1}{p}, (x, \alpha), \frac{1}{p})$ is at least the cardinality of the set $A_{n,p}$. Thus we have

$$h_d(f,(x,\alpha)) = \lim_{r\to 0}\lim_{p\to\infty}\varlimsup_{n\to\infty}\frac{1}{n}\ln N_d\Big(f,r,n,(x,\alpha),\frac{1}{p}\Big) \geqslant \lim_{p\to\infty}\varlimsup_{n\to\infty}\frac{1}{n}\ln N_d\Big(f,\frac{1}{p_0},n,(x,\alpha),\frac{1}{p}\Big) \geqslant \lim_{n\to\infty}\frac{(n-p)\ln 2}{n} = \ln 2. \quad \square$$

Completion of the proof of Theorem 2. Suppose that the function $(x, \alpha) \mapsto h_d(f, (x, \alpha))$ belongs to the first Baire class on the space $\Omega_2 \times \Omega_2$, then, by the Baire theorem on functions of the first class, in the space $\Omega_2 \times \Omega_2$ there must be points of continuity of the function $(x, \alpha) \mapsto h_d(f, (x, \alpha)$. On the other hand, the sets $\Omega_2 \times P_0$ and $\Omega_2 \times P_1$ are everywhere dense in the space $\Omega_2 \times \Omega_2$. Therefore, by virtue of Lemmas 1 and 2, each point of the space $\Omega_2 \times \Omega_2$ is a discontinuity point of the function $(x, \alpha) \mapsto h_d(f, (x, \alpha))$. Thus, the function $(x, \alpha) \mapsto h_d(f, (x, \alpha))$ is everywhere discontinuous and does not belong to the first Baire class on the space $\Omega_2 \times \Omega_2$. $\square$

References

[1] B. Hasselblatt and A. Katok, *A First Course in Dynamics. With a Panorama of Recent Developments.* Cambridge University Press, New York, 2003.

[2] A. N. Vetokhin, Baire classification of the local entropy considered as a function of a point in the phase space. (Russian) *Differ. Uravn.* **58** (2022), no. 11, 1573–1574.