Localization Property of Solutions for Parabolic PDE

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1 Introduction

We investigate the Cauchy–Dirichlet problem for a wide class of quasi-linear parabolic equations with the model representative:

$$u_t - \triangle u + g(t)|u|^{q-1}u = 0, \;\; 0 < q < 1,$$

where the continuous absorption potential $g(t)$ is positive for $t > 0$ and degenerates at $t = 0$: $g(0) = 0$. For an arbitrary boundary regime (without any subordination conditions), a certain type of weakened localization is obtained. Under some restriction from below on the degeneration of the potential, the strong localization holds for an arbitrary boundary regime (including regimes that do not satisfy any conditions of subordination).

It is well-known that, in case of non-degenerate absorption potential $g(t,x)$, i.e., when

$$g(t,x) \geq c_0 > 0 \;\; \forall\,(t,x) \in (0,T] \times \overline{\Omega},$$

an arbitrary energy solution of the considered problem has the finite-speed propagation property for solution's support:

$$\zeta(t) := \sup\big\{|x| :\; x \in \operatorname{supp} u(t,\,\cdot\,)\big\} < 1 + c(t), \;\; \text{where} \;\; c(t) \to 0 \;\; \text{as} \;\; t \to 0.$$

In particular, this implies the localization of solution (see, e.g., [3, 5] and the references therein):

$$\zeta(t) := \sup\big\{|x| :\;\; x \in \operatorname{supp} u(t,\,\cdot\,)\big\} < c_1 = c_1(T_1) < l \;\; \forall\, t :\; 0 \leq t < T_1 = T_1(l) \leq T. \tag{1.1}$$

For various quasi- and semi-linear parabolic equations, the localization of solutions' supports were studied by many authors (see, e.g., [3, 7] and the references therein). It seems that Kalashnikov [8] was the first who investigated the localization property for the first initial-boundary problem for a 1-D heat equation. More precisely, he considered problem (2.1)–(2.4) in the domain $[1, +\infty)$ with $n = 1$,

$$a_i(t,x,s,\xi) = \xi, \;\; \xi \in \mathbb{R}^1,$$
$$g(t,x) = g_0(t) \in C^1([1,+\infty)) \cap L_\infty([1,+\infty)), \;\; g_0(0) = 0, \;\; g_0(t) > 0 \;\; \forall\, t > 0$$

and

$$u(t,1) = f(t) \in C^1([1,+\infty)) \cap L_\infty([1,+\infty)).$$

Under the assumption

$$g_0(t)^{-1} \cdot f(t) \to 0 \ \text{ as } \ t \to 0, \tag{1.2}$$

he proved that solutions possess weak localization property for t separated from 0:

$$\sup \big\{ \zeta(t): \ 0 < \delta \le t < T \big\} < c_1 = c_1(\delta) < \infty \ \ \forall \, \delta > 0.$$

On the other side, following G. I. Barenblatt's conjecture on an initial jump of the free boundary, Kalashnikov in [8] proved that

$$\inf \big\{ \zeta(t): \ 0 < t < t_* \big\} \ge c_2 = c_2(t_*) > 0, \tag{1.3}$$

if potential $g_0(t)$ decreases fast enough when $t \to 0$. In particular, the free boundary has an initial jump (1.3), when

$$g_0(t) = t^{\frac{1}{2}} \exp\Big(-\frac{1}{t^2}\Big), \quad f_0(t) = t \exp\Big(-\frac{1}{t^2}\Big).$$

The analysis of [8] concerns only the case of strongly degenerating boundary regimes $f(t)$ (see condition (1.2)). Method [12] involutes arbitrary $f(t)$, which are strongly degenerate, weakly degenerate as well as non-degenerate as $t \to 0$. Also, note that the barrier technique of [8] can be applied only to equations that admit the comparison theorems. Our approach is adaptation and combination of a variant of local energy method and an estimate method of Saint–Venant's principle type. These methods are the result of a long evolution of ideas coming from the theory of linear elliptic and parabolic equations. The essence of the energy method consists of special inequalities links different energy norms of solutions. This method was developed and used in [2, 4, 5, 9, 11, 12]. The second approach is a technique of parameter's introduction. This method was offered by G. A. Iosif'jan and O. A. Oleinik [6].

2 Setting of the problem and the main results

Let $Q_T = (0,T) \times \Omega$, $0 < T < \infty$, $\Omega \subset \{x \in \mathbb{R}^n : \ x| > 1\}$ be a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with C^1-boundary $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega$, where

$$\partial_0\Omega = \big\{ x \in \mathbb{R}^n : \ |x| = 1 \big\}, \quad \partial_1\Omega \subset \big\{ x \in \mathbb{R}^n : \ |x| > l \big\}, \ \text{ where } \ l = const > 1. \tag{2.1}$$

The aim of this brief communication is to investigate the behavior of weak solutions of the following initial-boundary problem:

$$u_t - \sum_{i=1}^n \big(a_i(t,x,u,\nabla_x u) \big)_{x_i} + g(t,x)|u|^{q-1}u = 0 \ \text{ in } Q_T, \ \ 0 < q < 1, \tag{2.2}$$

$$u(t,x) = f(t,x) \ \text{ on } (0,T) \times \partial_0\Omega, \quad u(t,x) = 0 \ \text{ on } (0,T) \times \partial_1\Omega, \tag{2.3}$$

$$u(0,x) = 0 \ \ \forall \, x \in \Omega. \tag{2.4}$$

Here the functions $a_i(t,x,s,\xi)$ $(i = 1, \dots, n)$ are continuous in all arguments and satisfy the following conditions for $(t,x,s,\xi) \in (0,T) \times \Omega \times \mathbb{R}^1 \times \mathbb{R}^n$:

$$|a_i(t,x,s,\xi)| \le d_1|\xi|, \ \ d_1 = const < \infty,$$

$$\sum_{i=1}^n \big(a_i(t,x,s,\xi) - a_i(t,x,s,\eta) \big)(\xi_i - \eta_i) \ge d_0|\xi - \eta|^2, \ \ d_0 = const > 0.$$

The absorption potential $g(t,x)$ is continuous nonnegative function such that

$$g(t,x) > 0 \ \ \forall\,(t,x) \in (0,T] \times \overline{\Omega}; \quad g(0,x) = 0 \ \ \forall\ x \in \overline{\Omega}. \tag{2.5}$$

Without loss of generality assume that the function $f(t,x)$ in (2.3) is defined in the domain $(0,T)\times\Omega$ and

$$f(t,\,\cdot\,) \in L_2(0,T;H^1(\Omega,\partial_1\Omega)) \cap H^1(0,T;L_2(\Omega)).$$

Following [1], by a weak solution of problem (2.1)–(2.4) we understand the function

$$u(t,\,\cdot\,) \in f(t,\,\cdot\,) + L_2\big(0,T;H^1(\Omega,\partial\Omega)\big)$$

such that

$$u_t(t,\,\cdot\,) \in L_2(0,T;(H^1(\Omega,\partial\Omega))^*),$$

and u satisfies (2.3), (2.4) and the integral identity

$$\int\limits_{(0,T)} \langle u_t,\xi\rangle\,dt + \int\limits_{(0,T)\times\Omega} \sum_{i=1}^{n} a_i(t,x,u,\nabla_x u)\xi_{x_i}\,dx\,dt + \int\limits_{(0,T)\times\Omega} g(t,x)|u|^{q-1}u\xi\,dx\,dt = 0$$
$$\forall\,\xi \in L_2(0,T;H^1(\Omega,\partial\Omega)).$$

With boundary regime $f(t,x)$ from (2.3), we associate the function:

$$F(t) := \sup_{0\leqslant s\leqslant t} \int\limits_{\Omega} f(s,x)^2\,dx + \int\limits_0^t \int\limits_{\Omega} \big(|\nabla_x f|^2 + g(t,x)|f(t,x)|^{q+1}\big)\,dx\,dt + \int\limits_0^t \int\limits_{\Omega} |f_t(t,x)|^2\,dx\,dt \tag{2.6}$$

which will be used in all of our main results.

Theorem 1 (Theorem [Weakened localization for an arbitrary regime)**.** *Let the absorption potential g from* (2.2) *satisfy condition* (2.5)*. Then an arbitrary energy solution $u(t,x)$ to problem* (2.1)–(2.4) *possesses the weakened localization property. That is, there exists $\zeta_1(t) \in C(0,\infty)$ such that*

$$\zeta(t) \le \min(\zeta_1(t), cL_1) \ \ \textit{for all} \ \ t > 0,$$

where $\zeta(\,\cdot\,)$ is the compactification radius defined from (1.1) *and* $L_1 = \operatorname{diam}\Omega$.

The function $\zeta_1(t)$ may go to infinity as $t \to 0$. That is, an infinite initial jump of the support is possible.

Theorem 2 (Strong localization for an arbitrary regime)**.** *Let the function $F(\,\cdot\,)$ be from* (2.6)*, the absorption potential g from* (2.2) *have a nonnegative monotonic minorant:*

$$g(t,x) \ge g_\omega(t) := \exp\Big(-\frac{\omega(t)}{t}\Big) \ \ \forall\,t > 0,$$

where $\omega(t)$ is a nonnegative nondecreasing function such that $\omega(t) \to 0$ as $t \to 0$. Then an arbitrary energy solution $u(t,x)$ of problem (2.1)–(2.4) *possesses the strong localization property and the following upper estimate holds:*

$$\zeta(t) \le 1 + \frac{t}{2} + c_1\Big\{t\ln(c_2F(t)) + c_3 t\ln t^{-1} + c_4\omega\Big(\frac{t}{2}\Big)\Big\}^{\frac{1}{2}} \ \ \forall\,t < T.$$

Let us notice that in both theorems we do not impose any conditions on the function $F(\,\cdot\,)$ from (2.6).

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