Existence of Bounded Solutions of a Dynamic Equation

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1 Basic concepts of the theory of time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real line $\mathbb{R}$. Assume $\mathbb{T}$ has the topology that it inherits from $\mathbb{R}$ with the standard topology.

Since the object of our study is the oscillations of solutions of dynamic equations, we will assume $\sup\mathbb{T}=\infty$. For any interval $[a,b]\subset\mathbb{R}$ we define $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$.

For a time scale $\mathbb{T}$, the *forward jump operator* $\sigma(t):\mathbb{T}\to\mathbb{T}$ is defined as $\sigma(t)=\inf\{s\in\mathbb{T}:\ s>t\}$; the *backward jump operator* $\rho(t):\mathbb{T}\to\mathbb{T}$ is defined as $\rho(t)=\sup\{s\in\mathbb{T}:\ s<t\}$. The *graininess function* $\mu:\mathbb{T}\to[0,1)$ is defined as $\mu(t):=\sigma(t)-t$.

A point $t\in\mathbb{T}$ is called *right-dense* if $t>\inf\mathbb{T}$ and $\rho(t)=t$. A point $t\in\mathbb{T}$ is called *left-dense* if $t<\sup\mathbf{T}$ and $\sigma(t)=t$. Points that are right- and left-dense at the same time are called *dense*.

If $\sigma(t)>t$ ($\rho(t)<t$), we say that t is *right-scattered* (*left-scattered*). Points that are right- and left-scattered at the same time are called *isolated points*.

If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k=\mathbb{T}\setminus\{M\}$; otherwise, $\mathbb{T}^k=\mathbb{T}$.

A function $f:\mathbb{T}\to\mathbb{R}^d$ is called Δ-*differentiable* at $t\in\mathbb{T}^k$ if there exists the finite in $\mathbb{R}^d$ limit

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(t)}{\sigma-t}\,,$$

and the number $f^{\Delta}(t)$ is called the Δ-*derivative* at the point t.

We cite some known results [1]:

(a) If $t\in\mathbb{T}^k$ is a right-dense point of a time scale $\mathbb{T}$, then f is Δ-differentiable at t iff the limit

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists in $\mathbb{R}^d$.

(b) If $t\in\mathbb{T}^k$ is a right-scattered point of a time scale $\mathbb{T}$ and f is continuous at t, then f is Δ-differentiable at t and

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}\,.$$

2 Problem statement and auxiliary results

We consider the system of differential equations

$$\frac{dx}{dt} = X(t,x), \tag{2.1}$$

where $x \in D$, $D \subset \mathbb{R}^d$, and the corresponding system of dynamic equations

$$x_\lambda^\Delta = X(t, x_\lambda), \tag{2.2}$$

where $t \in \mathbb{T}_\lambda$, $\lambda \in \Lambda \subset \mathbb{R}$, $\lambda = 0$ is a limit point of the set Λ, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^d$, and $x_\lambda^\Delta(t)$ is the Δ-derivative of $x_\lambda(t)$ in $\mathbb{T}_\lambda$.

Assume that $X(t,x)$ is continuously differentiable and bounded with its partial derivatives, i.e. there exists $C > 0$ such that

$$|X(t,x)| + \left|\frac{\partial X(t,x)}{\partial t}\right| + \left\|\frac{\partial X(t,x)}{\partial x}\right\| \le C \tag{2.3}$$

for $t \in \mathbb{T}_\lambda$ and $x \in D$. Here $\frac{\partial X}{\partial x}$ is the corresponding Jacobian matrix, $|\cdot|$ is the Euclidian norm of a vector, and $\|\cdot\|$ is the norm of a matrix.

Let $\mu_\lambda := \sup\limits_{t\in\mathbb{T}_\lambda} \mu_\lambda(t)$, where $\mu_\lambda : \mathbb{T}_\lambda \to [0,\infty)$ is the graininess function. If $\mu_\lambda \to 0$ as $\lambda \to 0$, then $\mathbb{T}_\lambda$ approaches the continuous time scale $\mathbb{T}_0 = \mathbb{R}$. Therefore, it is natural to expect that, under certain conditions, the existence of a bounded solution of equation (2.1) implies the existence of a bounded solution of equation (2.2) on the time scale $\mathbb{T}_\lambda$.

Let $t_0, t_0 + T \in \mathbb{T}_\lambda$, and let $x(t)$ and $x_\lambda(t)$ be solutions of (2.1) and (2.2) on $[t_0, t_0 + T]$ and on $[t_0, t_0 + T]_{\mathbb{T}_\lambda}$, respectively, with initial conditions $x(t_0) = x_0$, $x_\lambda(t_0) = x_{\lambda 0}$.

Lemma 2.1 ([3])**.** *If x_λ and $x(t)$ are the corresponding solutions of* (2.2) *and* (2.1)*, then the inequality*

$$|x(t) - x_\lambda(t)| \le \mu(\lambda) K(T) \tag{2.4}$$

holds for $t \in [t_0, t_0 + T]_{\mathbb{T}_\lambda}$. Here

$$\mu(\lambda) = \sup_{t\in[t_0,t_0+T]_{\mathbb{T}_\lambda}} \mu_\lambda(t), \quad K(T) = \max\{L_1, L_2\},$$

$$L_1 = \mu_\lambda\Big(\Pi e^C C_1 + \frac{1}{4}\,\Pi C_1^2 e^C\Big), \quad L_2 = \mu_\lambda\Big(\Pi e^C\Big(C_1 + \frac{C_1^2}{4}\Big) + 3C_1\Big).$$

Under condition (2.3), the following statement holds.

Lemma 2.2 ([3])**.** *A solution x_λ of system* (2.2) *continuously depends on the initial data until the moment it leaves the region D.*

We also give the definition of the exponential stability for solutions of dynamic equations on time scales which is similar to the definition of the exponential stability for solutions of differential equations [2].

Definition 2.1. A solution $x_\lambda(t)$ of system (2.2), defined on $\mathbb{T}_\lambda$, is called exponentially stable, uniformly in t_0, if there exist $\delta > 0$, $N > 0$ and $\alpha > 0$ such that for any solution $y_\lambda(t)$ of system (2.2), satisfying

$$|x_\lambda(t_0) - y_\lambda(t_0)| < \delta,$$

the inequality

$$|x_\lambda(t) - y_\lambda(t)| \le N e^{-\alpha(t-t_0)}\big|x_\lambda(t_0) - y_\lambda(t_0)\big|$$

holds for $t \ge t_0$. Here the constants δ, N and α are independent of t_0.

3 Main results

We found the minimum conditions on the graininess function μ_0 under which the existence of a bounded solution of the dynamical system (2.2) on the corresponding time scale $\mathbb{T}_{\lambda_0}$ implies the existence of a bounded solution of this system on any scale whose graininess function is less than μ_0.

Theorem 3.1. *Let the following conditions be satisfied:*

(1) *$X(t,x)$ is defined and continuously differentiable for $t \in \mathbb{R}$, $x \in D$, where D is a domain in the space $\mathbb{R}^d$, and satisfies condition* (2.3)*.*

(2) *There exists $\mu_0 > 0$ such that system* (2.2) *has a bounded on $\mathbb{T}_{\lambda_0}$ and exponentially stable, uniformly in t_0, solution $x_{\lambda_0}(t)$, which belongs to D together with some its ρ-neighborhood.*

Then, if the inequalities

$$\mu_0 K\Big(\frac{\ln 4N}{\alpha} + 1\Big) \le \frac{\delta}{8}, \tag{3.1}$$

$$\frac{3N\delta}{2} < \rho, \tag{3.2}$$

$$\mu_0 \le \frac{\rho}{4C} \tag{3.3}$$

hold, where δ, N and α are the constants from Definition 2.1 and C is from condition (2.3)*, then, for all μ_λ satisfying $\mu_\lambda < \mu_0$, system* (2.2) *has a solution bounded on $\mathbb{T}_\lambda$.*

Proof. Without loss of generality, we set $t_0 = 0$ and $x_\lambda(0) = x(0)$.

It follows from condition (2) of this theorem that, for $\mu_\lambda = \mu_0$, system (2.2) has an exponentially stable, uniformly in t_{k_0}, solution x_{λ_0}, which belongs to D together with some its ρ-neighborhood. Hence, there exists a constant $C_0 > 0$ such that

$$|x_{\lambda_0}(t_k)| \le C_0 \ \text{ for an arbitrary } t_k \in \mathbb{T}_{\lambda 0}.$$

Let t_{k_0} be the smallest number on the time scale $\mathbb{T}_{\lambda_0}$, defined by the graininess function μ_0, such that $t_{k_0} \ge \frac{\ln 4N}{\alpha}$. Clearly, $t_{k_0} \le \frac{\ln 4N}{\alpha} + 1$.

Now we fix $0 < \mu_\lambda < \mu_0$ and denote by x_λ solutions of system (2.2) on the corresponding time scale $\mathbb{T}_\lambda$.

Let us consider points $t \in [0, t_{k_0}]_{\mathbb{T}_\lambda}$. For every t one can indicate the smallest number t_k on the scale $\mathbb{T}_{\lambda_0}$ such that

$$0 \le t_k - t \le \mu_0.$$

Let x_λ be a solution of system (2.2) such that $x_\lambda(0) = x_{\lambda_0}(0)$. We denote by $x(t)$ the solution of system (2.1) with the initial data $x(0) = x_{\lambda_0}(0)$. We can show that $x(t)$ can be continued to the interval $[0, t_{k_0}]$. Indeed, in view of inequality (3.3) and Lemma 2.1, it follows from Picard's theorem that $x(t)$ is defined at each point $n\mu_0 \le t_{k_0}$, $n \in \mathbb{N}$, and takes on the values which belong to D together with their $\frac{\rho}{2}$-neighborhoods. Thus, the solution $x(t)$ is continued to the whole interval $[0, t_{k_0}]$ and belongs to D together with its $\frac{\rho}{2}$-neighborhood. It follows from (2.4) and (3.1) that the solution x_λ of system (2.2) is defined for all $t \in \mathbb{T}_\lambda$ that do not exceed $t_{k_0} \in \mathbb{T}_{\lambda_0}$, and belongs to the domain D.

Further, we partition the axis into the intervals $[nt_{k_0}, (n+1)t_{k_0}]$ and denote by t_n the largest numbers in $\mathbb{T}_\lambda$ such that $t_n \le nt_{k_0}$. Let us examine how the solution $x_\lambda(t)$ of equation (2.2), starting at t_n, $n \in \mathbb{N}$, behaves on $[t_n, (n+1)t_{k_0})]_{\mathbb{T}_\lambda}$.

Let us now construct a solution of equation (2.2) which is bounded on the whole axis of the timescale $\mathbb{T}_\lambda$.

Let x_{λ,t^*} be such a solution of equation (2.2) which starts at a point t^* of $\mathbb{T}_\lambda$, $t \geq t^*$, and $x_{\lambda,t^*}(t^*) = x_\lambda(t^*)$.

For each t^* we choose the smallest non-negative number $\tilde{t}_{\lambda_0} \in \mathbb{T}_{\lambda_0}$ such that $t^* \leq \tilde{t}_{\lambda_0} \leq t^* + \mu_0$.

We now consider a solution $x_{\lambda,t^*}(t)$ such that $|x_{\lambda,t^*}(t^*) - x_{\lambda_0}(\tilde{t}_{\lambda_0})| \leq \frac{3\delta}{4}$, where x_{λ_0} is a bounded solution of equation (2.2) on the timescale $\mathbb{T}_{\lambda_0}$ with the graininess function $\mu_\lambda = \mu_0$, which is indicated in the statement of this theorem.

We partition the left semi-axis of the timescale $\mathbb{T}_\lambda$ into the intervals $[-nt_{k_0}, -(n+1)t_{k_0}]$, $n \to -\infty$. For each point $-nt_{k_0}$ we choose the largest $t_n \in \mathbb{T}_\lambda$ such that

$$t_n \leq -nt_{k_0} \leq t_n + \mu_0.$$

The point t_0 is chosen in the same way.

Let us now consider the set of solutions x_{λ,t_n} of equation (2.2), whose initial data satisfy the inequality

$$\left|x_{\lambda,t_n}(t_n) - x_{\lambda_0}(-nt_{k_0})\right| \leq \frac{3\delta}{4}.$$

Obviously, these solutions satisfy conditions 1°–3°. Let S_n be the set of values of these solutions at t_n. Each S_n is the image of the ball of radius $\frac{3\delta}{4}$ centered at the point $x_{\lambda_0}(-nt_{k_0})$, generated by the mapping x_{λ,t_n}. By Lemma 2.2 and conditions 1°-3°, each set S_n is a nonempty subset of S_{n-1} and a compact.

Let us denote $z = \bigcap_n S_n$ and consider the solution x_{λ,t_1} of equation (2.2) with the initial condition $x_{\lambda,t_1}(t_1) = z$. This solution can be continued to the left to the point t_n, at which it belongs to the $\frac{3\delta}{4}$-neighborhood of $x_{\lambda_0}(t_n)$ for every natural n. It means that this solution is defined for all t satisfying the inequality in 3°. Hence, it is bounded. This proves that system (2.2) has a bounded solution, defined on $\mathbb{T}_\lambda$. □

The following statement provides the conditions for the existence of a solution of system (2.2) bounded on $\mathbb{T}_\lambda$ given the existence of such a solution of the corresponding system (2.1).

Theorem 3.2. *Let the following conditions be satisfied:*

(1) *$X(t,x)$ is defined and continuously differentiable for $t \in \mathbb{R}$, $x \in D$, where D is a domain in $\mathbb{R}^d$, and satisfies condition* (2.3).

(2) *System* (2.1) *has a bounded on $\mathbb{R}$ and exponentially stable, uniformly in $t_0 \in \mathbb{R}$, solution $x(t)$, which belongs to D together with some its ρ-neighborhood.*

Then there exists μ_0 such that for all $0 < \mu_\lambda \leq \mu_0$ system (2.2) *has a solution $x_\lambda(t)$ bounded on $\mathbb{T}_\lambda$. Moreover,*

$$\sup_{t\in\mathbb{T}_\lambda} |x_\lambda(t) - x(t)| \to 0, \quad \mu_\lambda \to 0.$$

The existence of $\mu_0 > 0$, such that for all $0 < \mu_\lambda \leq \mu_0$ system (2.2) has a solution $x_\lambda(t)$ bounded on $\mathbb{T}_\lambda$, follows from Theorem 2.3 [3].

We also obtained the opposite result.

Theorem 3.3. *Let the following conditions be satisfied:*

(1) *the function $X(t,x)$ satisfies condition (1) of Theorem 3.1;*

(2) *there exists* $\mu_0 > 0$ *such that system* (2.2) *with initial data at the point* $t_0 = 0$ *has a solution, bounded on* $\mathbb{T}_{\lambda_0}$ *and uniformly in* k_0 *exponentially stable, which belongs to* D *with some its* ρ*-neighborhood.*

Then, if inequalities (3.1)–(3.3) *hold, then system* (2.1) *has a solution bounded on* $\mathbb{R}$.

The proof of this theorem is based on the reasoning in the proof of Theorem 3.1.

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