Existence of Solutions to BVPs at Resonance for Mixed Caputo Fractional Differential Equations

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1 Introduction

Let $J=[0,1]$, $X=C(J)\times\mathbb{R}$ and $\|x\|=\max\{|x(t)|:\ t\in J\}$ be the norm in $C(J)$.

We discuss the fractional boundary value problem

$$ {}^c\!\mathcal{D}^{\alpha}_{1-}\,{}^c\!\mathcal{D}^{\beta}_{0+}x(t)=f(t,x(t)), \tag{1.1}$$

$$ u(0)={}^c\!\mathcal{D}^{\beta}_{0+}x(t)\Big|_{t=0}={}^c\!\mathcal{D}^{\beta}_{0+}x(t)\Big|_{t=1}, \tag{1.2}$$

where $\alpha,\beta\in(0,1)$, $f\in C(J\times\mathbb{R})$, ${}^c\!\mathcal{D}_{1-}$ and ${}^c\!\mathcal{D}_{0+}$ denote the right and the left Caputo fractional derivatives.

Definition 1.1. We say that $x:J\to\mathbb{R}$ is *a solution of equation* (1.1) if $x,{}^c\!\mathcal{D}^{\beta}_{0+}x\in C(J)$ and x satisfies (1.1) for $t\in J$. A solution x of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

Let $x:J\to\mathbb{R}$, $\gamma\in(0,1)$ and $\mu\in(0,\infty)$. Then the left ${}^c\!\mathcal{D}^{\gamma}_{0+}x$ and the right ${}^c\!\mathcal{D}^{\gamma}_{1-}x$ Caputo fractional derivatives of x of order γ are defined respectively by [2, 3]

$$ {}^c\!\mathcal{D}^{\gamma}_{0+}x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^t\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(0))\,\mathrm{d}s$$

and

$$ {}^c\!\mathcal{D}^{\gamma}_{1-}x(t)=-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_t^1\frac{(s-t)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(1))\,\mathrm{d}s,$$

where Γ is the Euler gamma function.

The left $I^{\mu}_{0+}x$ and the right $I^{\mu}_{1-}x$ Riemann–Liouville fractional integrals of x of order μ are defined respectively by

$$ I^{\mu}_{0+}x(t)=\int\limits_0^t\frac{(t-s)^{\mu-1}}{\Gamma(\mu)}\,x(s)\,\mathrm{d}s\ \text{ and }\ I^{\mu}_{1-}x(t)=\int\limits_t^1\frac{(s-t)^{\mu-1}}{\Gamma(\mu)}x(s)\,\mathrm{d}s.$$

If $x\in C(J)$ and $\gamma\in(0,1)$, then

$$ \begin{gathered} {}^c\!\mathcal{D}^{\gamma}_{0+}I^{\gamma}_{0+}x(t)=x(t),\quad {}^c\!\mathcal{D}^{\gamma}_{1-}I^{\gamma}_{1-}x(t)=x(t)\ \text{ for }\ t\in J,\\ I^{\gamma}_{0+}\,{}^c\!\mathcal{D}^{\gamma}_{0+}x(t)=x(t)-x(0),\quad I^{\gamma}_{1-}\,{}^c\!\mathcal{D}^{\gamma}_{1-}x(t)=x(t)-x(1)\ \text{ for }\ t\in J \end{gathered}$$

and

$$I_{0+}^{\gamma_1} I_{0+}^{\gamma_2} x(t) = I_{0+}^{\gamma_1+\gamma_2} x(t), \quad I_{1-}^{\gamma_1} I_{1-}^{\gamma_2} x(t) = I_{1-}^{\gamma_1+\gamma_2} x(t) \text{ for } t \in J, \ \ \gamma_1, \gamma_2 \in (0,\infty).$$

Problem (1.1), (1.2) is at resonance because $\left\{c(1+\frac{t^\beta}{\Gamma(\beta+1)}) : c \in \mathbb{R}\right\}$ is the set of nontrivial solutions to the homogeneous boundary value problem ${}^c\!D_{1-}^{\alpha}\, {}^c\!D_{0+}^{\beta} x = 0$, (1.2).

2 Operator $\mathcal{H}$ and its properties

Let an operator $\mathcal{H} : X \to X$ be given by the formula

$$\mathcal{H}(x,c) = \left(c\Big(1 + \frac{t^\beta}{\Gamma(\beta+1)}\Big) + I_{0+}^{\beta} I_{1-}^{\alpha} f(t,x(t)), c - I_{1-}^{\alpha} f(t,x(t))\Big|_{t=0} \right).$$

Note that

$$I_{0+}^{\beta} I_{1-}^{\alpha} f(t,x(t)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(\int_s^1 \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau))\,\mathrm{d}\tau \right) \mathrm{d}s$$

and

$$I_{1-}^{\alpha} f(t,x(t))\big|_{t=0} = \int_0^1 \frac{s^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\,\mathrm{d}s.$$

If $x \in C(J)$ and $0 \le t_1 < t_2 \le 1$, then

$$\left| I_{0+}^{\beta} I_{1-}^{\alpha} x(t) \right| \le \frac{\|x\|}{\Gamma(\beta+1)\Gamma(\alpha+1)}, \quad t \in J, \tag{2.1}$$

$$\left| I_{0+}^{\beta} I_{1-}^{\alpha} x(t)\big|_{t=t_2} - I_{0+}^{\beta} I_{1-}^{\alpha} x(t)\big|_{t=t_1} \right| \le \frac{2\|x\|}{\Gamma(\beta+1)\Gamma(\alpha+1)} (t_2-t_1)^\beta.$$

Lemma 2.1. *$\mathcal{H}$ is a completely continuous operator.*

The following result gives the relation between fixed points of $\mathcal{H}$ and solutions to problem (1.1), (1.2).

Lemma 2.2. *If $(x,c) \in X$ is a fixed point of $\mathcal{H}$, then x is a solution of problem* (1.1), (1.2).

Proof. Let $\mathcal{H}(x,c) = (x,c)$ for some $(x,c) \in X$. Then

$$x(t) = c\Big(1 + \frac{t^\beta}{\Gamma(\beta+1)}\Big) + I_{0+}^{\beta} I_{1-}^{\alpha} f(t,x(t)), \ \ t \in J, \tag{2.2}$$

$$I_{1-}^{\alpha} f(t,x(t))\Big|_{t=0} = 0. \tag{2.3}$$

Applying ${}^c\!D_{0+}^{\beta}$ to (2.2), we get

$$ {}^c\!D_{0+}^{\beta} x(t) = c + I_{1-}^{\alpha} f(t,x(t)), \ \ t \in J. \tag{2.4}$$

Hence ${}^c\!D_{0+}^{\beta} x \in C(J)$, ${}^c\!D_{0+}^{\beta} x(t)|_{t=1} = c$ and (see (2.3)) ${}^c\!D_{0+}^{\beta} x(t)|_{t=0} = c$. We now apply ${}^c\!D_{1-}^{\alpha}$ to (2.4) and have

$$ {}^c\!D_{1-}^{\alpha}\, {}^c\!D_{0+}^{\beta} x(t) = f(t,x(t)), \ \ t \in J.$$

Thus x is a solution of equation (1.1). From

$$ {}^c\!D_{0+}^{\beta} x(t)\Big|_{t=1} = c, \quad {}^c\!D_{0+}^{\beta} x(t)\Big|_{t=0} = c$$

and (see (2.2)) $x(0) = c$ it follows that x satisfies (1.2). Consequently, x is a solution of problem (1.1), (1.2). $\square$

3 Existence result

Theorem 3.1. *Suppose that*

(H_1) *there exists $M > 0$ such that $xf(t,x) > 0$ for $t \in J$ and $|x| \ge M$;*

(H_2) *there exist positive constants A, B and $\rho \in (0,1)$ such that $|f(t,x)| \le A + B|x|^\rho$ for $t \in J$ and $x \in \mathbb{R}$.*

Then problem (1.1), (1.2) *has at least one solution.*

Proof. Keeping in mind Lemma 2.2, we need to prove that $\mathcal{H}$ admits a fixed point. We prove the existence of a fixed point of $\mathcal{H}$ by the Schaefer fixed point theorem [1, 4]. To this end, let

$$\Omega = \Big\{(x,c) \in X : \ \ (x,c) = \lambda\mathcal{H}(x,c) \ \text{ for some } \ \lambda \in (0,1)\Big\}.$$

Since $\mathcal{H}$ is a completely continuous operator, it follows from the Schaefer fixed point theorem that the boundedness of Ω in X guarantees the existence of a fixed point of $\mathcal{H}$.

Let $(x,c) = \lambda\mathcal{H}(x,c)$ for some $(x,c) \in X$ and $\lambda \in (0,1)$, that is,

$$x(t) = \lambda c\Big(1 + \frac{t^\beta}{\Gamma(\beta+1)}\Big) + \lambda I^\beta_{0+} I^\alpha_{1-} f(t,x(t)), \ \ t \in J, \tag{3.1}$$

$$(1-\lambda)c = -\lambda I^\alpha_{1-} f(t,x(t))\Big|_{t=0}. \tag{3.2}$$

We claim that

$$|x(\xi)| < M \ \text{ for some } \ \xi \in J, \tag{3.3}$$

where M is from (H_1). By (3.1), $x(0) = \lambda c$. Suppose that $x > M$ on J. Then $c > 0$ and, by (H_1), $I^\alpha_{1-} f(t,x(t))|_{t=0} > 0$, contrary to (3.2) because $(1-\lambda)c > 0$ and $I^\alpha_{1-} f(t,x(t))|_{t=0} > 0$. Similarly, $x < -M$ on J gives contrary to (3.2). Hence (3.3) is valid.

Putting $t = \xi$ in (3.1), we have

$$\lambda c = \frac{1}{1 + \xi^\beta/\Gamma(\beta+1)}\,\Big(x(\xi) - \lambda I^\beta_{0+} I^\alpha_{1-} f(t,x(t))\big|_{t=\xi}\Big). \tag{3.4}$$

Thus (see (3.1))

$$x(t) = \frac{1 + t^\beta/\Gamma(\beta+1)}{1 + \xi^\beta/\Gamma(\beta+1)}\,\Big(x(\xi) - \lambda I^\beta_{0+} I^\alpha_{1-} f(t,x(t))\big|_{t=\xi}\Big) + \lambda I^\beta_{0+} I^\alpha_{1-} f(t,x(t)), \ \ t \in J.$$

Hence (see (H_2), (2.1) and (3.3))

$$|x(t)| \le \Big(1 + \frac{1}{\Gamma(\beta+1)}\Big)\Big(M + \frac{A + B\|x\|^\rho}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big) + \frac{A + B\|x\|^\rho}{\Gamma(\beta+1)\Gamma(\alpha+1)}\,, \ \ t \in J.$$

In particular,

$$\|x\| \le W_1 + W_2\|x\|^\rho, \tag{3.5}$$

where

$$W_1 = M\Big(1 + \frac{1}{\Gamma(\beta+1)}\Big) + \frac{A}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big(2 + \frac{1}{\Gamma(\beta+1)}\Big),$$
$$W_2 = \frac{B}{\Gamma(\beta+1)\Gamma(\alpha+1)}\,\Big(2 + \frac{1}{\Gamma(\beta+1)}\Big).$$

Since

$$\lim_{v\to\infty}\frac{W_1+W_2v^\rho}{v}=0,$$

there exists $S>0$ such that $W_1+W_2v^\rho<v$ or $v>S$. Consequently (see (3.5)), $\|x\|\le S$.

Hence $|f(t,x(t))|\le L$ for $t\in J$, where $L=A+BS^\rho$. In order to give the bound for c, we consider two cases if $\lambda\in(0,1/2]$ of $\lambda\in(1/2,1)$. Let $\lambda\in(0,1/2]$. Then (see (3.2))

$$|c|\le\frac{\lambda}{1-\lambda}\int_0^1\frac{s^{\alpha-1}}{\Gamma(\alpha)}\,|f(s,x(s))|\,\mathrm{d}s\le\frac{L}{\Gamma(\alpha+1)}\,.$$

Let $\lambda\in(1/2,1)$. Then (see (3.4))

$$|c|\le\frac{1}{\lambda(1+\xi^\beta/\Gamma(\beta+1))}\Big(|x(\xi)|+\big|I^\beta_{0+}I^\alpha_{1-}f(t,x(t))|_{t=\xi}\big|\Big)\le 2\Big(M+\frac{L}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big).$$

To summarize, we have $|c|\le D$, where

$$D=\max\left\{\frac{L}{\Gamma(\alpha+1)}\,,2\Big(M+\frac{L}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big)\right\}.$$

As a result, Ω is bounded and $\|x\|\le S$, $|c|\le D$ for $(x,c)\in\Omega$. □

Example 3.2. Let $p\in C(J)$, $\rho\in(0,1)$ and $f(t,x)=p(t)+\sin x+2|x|^\rho\arctan x$. Then f satisfies conditions (H_1) and (H_2) for $M=\sqrt[\rho]{1+\|p\|}$ and $A=1+\|p\|$, $B=\pi$. By Theorem 3.1, there exists a solution x of the equation

$${}^c\!D^\alpha_{1-}\,{}^c\!D^\beta_{0+}x(t)=p(t)+\sin x(t)+2|x(t)|^\rho\arctan x(t),$$

satisfying the boundary condition (1.2).

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