Bifurcation of Positive Periodic Solutions to Non-Autonomous Undamped Duffing Equations

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The extended abstract concerns the parameter-dependent periodic problem

$$u'' = p(t)u - h(t)|u|^{\lambda} \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega), \tag{1}$$

where $p, h, f \in L([0,\omega])$, $h \geq 0$ a.e. on $[0,\omega]$, $\lambda > 1$, and $\mu \in \mathbb{R}$ is a parameter. By a solution to problem (1), as usual, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets the periodic conditions. The text is based on the paper [3].

We first note that the differential equation in (1) with $\lambda = 3$ is derived, for example, when approximating non-linearities in the equations of motion of the oscillators in Figs. 1 and 2.

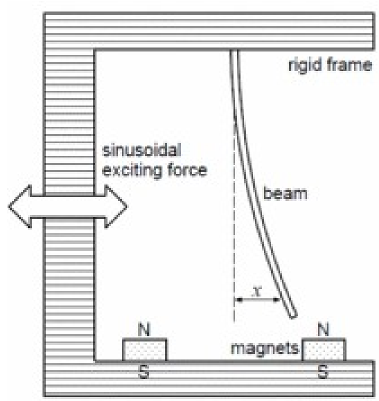

Figure 1. Forced steel beam deflected toward the two magnets[1].

Consider a forced undamped oscillator consisting of a mass body of weight m and a linear spring of characteristic k and non-deformed length ℓ (see Fig. 2). Assume that the mass body moves horizontally without any friction and the spring's base point B oscillates vertically, i.e., d is a positive ω-periodic function. This is a system with a single degree of freedom, described by the coordinate x, whose equation of motion is of the form

$$x'' = \frac{k}{m}\, x\left(\frac{\ell}{\sqrt{d^2(t) + x^2}} - 1\right) + \frac{F(t)}{m}\,. \tag{2}$$

A classical approach to deriving Duffing equation is to approximate the non-linearity in (2) by a third-degree Taylor polynomial centred at 0. We thus get the equation

$$x'' = \frac{k(\ell - d(t))}{md(t)}\, x - \frac{k\ell}{2md^3(t)}\, x^3 + \frac{F(t)}{m}\,, \tag{3}$$

[1]A figure is adopt from `http://www.scholarpedia.org/article/Duffing_oscillator`.

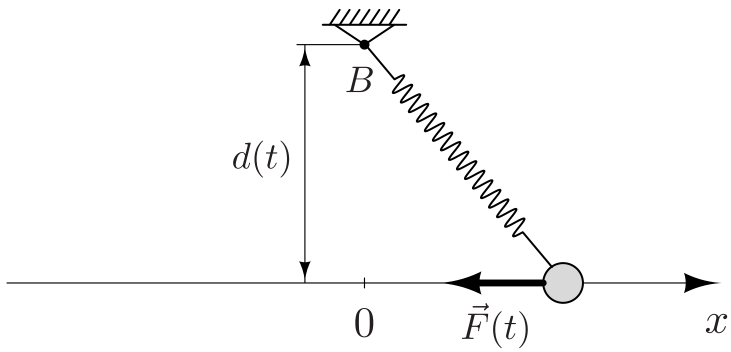

Figure 2. Forced undamped mass-spring oscillator with the so-called geometric non-linearity.

which is a particular case of the differential equation in (1). It is worth mentioning that the results below can be applied, for instance, to the forcing terms

$$F(t) := -f_0, \quad F(t) := A\Big(\sin\frac{2\pi t}{\omega} - \frac{1}{2}\Big),$$

where $f_0, A > 0$ are parameters.

To formulate our results, we need the following definitions.

Definition 1 ([2, Definitions 0.1 and 15.1, Proposition 15.2])**.** We say that a function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^-(\omega)$ if, for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a.e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) \geq u'(\omega),$$

the inequality $u(t) \leq 0$ holds for $t \in [0,\omega]$.

Remark 1. Let $\omega > 0$. If $p(t) := p_0$ for $t \in [0,\omega]$, then one can show by direct calculation that $p \in \mathcal{V}^-(\omega)$ if and only if $p_0 > 0$. For non-constant functions $p \in L([0,\omega])$, efficient conditions guaranteeing the inclusion $p \in \mathcal{V}^-(\omega)$ are provided in [2] (see also [1,4]).

Definition 2 ([2, Definition 16.1])**.** Let $p, f \in L([0,\omega])$. We say that a pair (p,f) belongs to the set $\mathcal{U}(\omega)$, if the problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a unique solution which is positive.

Remark 2. Let $p \in \mathcal{V}^-(\omega)$. It follows from [2, Theorem 16.2] that $(p,f) \in \mathcal{U}(\omega)$ provided that

$$\int\limits_0^\omega [f(s)]_- \,\mathrm{d}s > \mathrm{e}^{\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+ \,\mathrm{d}s} \int\limits_0^\omega [f(s)]_+ \,\mathrm{d}s. \tag{4}$$

In particular, if

$$f(t) \leq 0 \quad \text{for a.e. } t \in [0,\omega], \;\; f(t) \not\equiv 0, \tag{5}$$

then $(p,f) \in \mathcal{U}(\omega)$.

In what follows, we discuss the existence/non-existence as well as the exact multiplicity of positive solutions to problem (1) depending on the choice of the parameter μ provided that $p \in \mathcal{V}^-(\omega)$. Let us show, as a motivation, what happens in the autonomous case of (1). Hence, consider the equation

$$x'' = ax - b|x|^\lambda \operatorname{sgn} x - \mu. \tag{6}$$

In view of Remark 1 and the hypothesis $h \geq 0$ a.e. on $[0, \omega]$, we assume that $a, b > 0$. By direct calculation, the phase portraits of equation (6) can be elaborated depending on the choice of the parameter $\mu \in \mathbb{R}$ (see, Fig. 3) and, thus, one can prove the following proposition concerning the positive periodic solutions to equation (6).

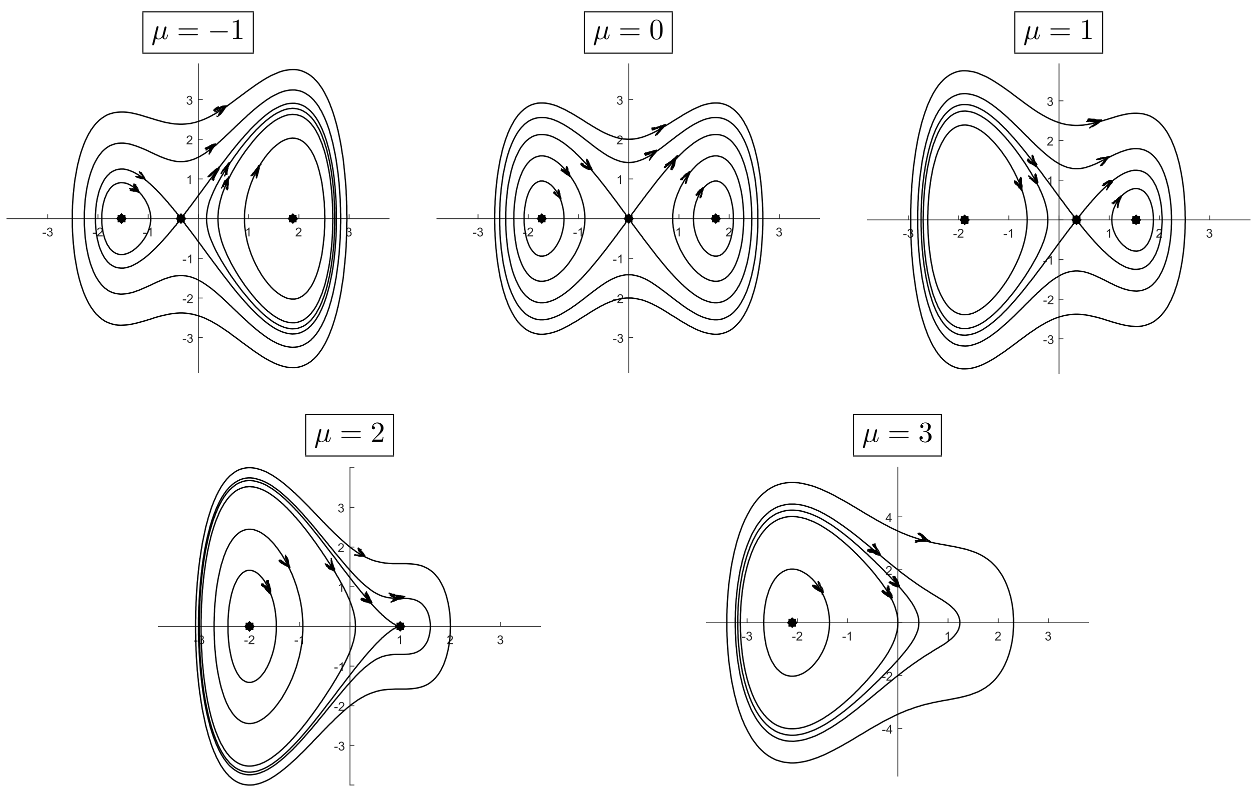

Figure 3. Phase portraits of equation (6) with $a = 3$, $b = 1$, and $\lambda = 3$.

Proposition 1. *Let $\lambda > 1$ and $a, b > 0$. Then, the following conclusions hold:*

(1) *If $\mu \leq 0$, then equation (6) has a unique positive equilibrium (center) and non-constant positive periodic solutions with different periods.*

(2) *If $0 < \mu < \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (6) possesses exactly two positive equilibria $x_2 > x_1$ (x_1 is a saddle and x_2 is a center) and non-constant positive periodic solutions with different periods. Moreover, all the non-constant positive periodic solutions are greater than x_1 and oscillate around x_2.*

(3) *If $\mu = \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (6) has a unique positive equilibrium (cusp) and no non-constant positive periodic solution occurs.*

(4) *If $\mu > \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (6) has no positive periodic solution.*

Proposition 1 shows that, if we consider μ as a bifurcation parameter, then, crossing the value $\mu^* = \frac{(\lambda-1)a}{\lambda}\,(\frac{a}{\lambda b})^{\frac{1}{\lambda-1}}$, a bifurcation of positive periodic solutions to equation (6) occurs. In Fig. 3, the critical value of the bifurcation parameter is $\mu^* = 2$.

Theorem 1 (Main result)**.** *Let $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, $(p, f) \in \mathcal{U}(\omega)$, $\int\limits_0^\omega f(s)\,\mathrm{d}s < 0$, and*

$$h(t) \geq 0 \quad \textit{for a. e.}\ \ t \in [0, \omega], \ \ h(t) \not\equiv 0. \tag{7}$$

Then, there exists $\mu_0 \in \,]0, +\infty[$ such that the following conclusions hold:

(1) *If $\mu = 0$, then problem (1) has at least one positive solution and, for any couple of distinct positive solutions u_1, u_2 to (1), the conditions*

$$\min\big\{u_1(t) - u_2(t) : \ t \in [0, \omega]\big\} < 0, \quad \max\big\{u_1(t) - u_2(t) : \ t \in [0, \omega]\big\} > 0$$

hold. If, moreover,

$$\mathrm{e}^{-1+\sqrt{1+\omega\int\limits_0^\omega p(s)\,\mathrm{d}s}}\,\bigg(-1 + \sqrt{1 + \omega\int\limits_0^\omega p(s)\,\mathrm{d}s}\bigg) \leq \frac{8}{\lceil\lambda\rceil}\,, \tag{8}$$

where $\lceil\,\cdot\,\rceil$ is the ceiling function, then problem (1) with $\mu = 0$ has a unique positive solution.

(2) *If $0 < \mu < \mu_0$, then problem (1) has solutions u_1, u_2 such that*

$$u_2(t) > u_1(t) > 0 \quad \textit{for}\ \ t \in [0, \omega]$$

and every non-negative solution u to problem (1) different from u_1 and u_2 satisfies

$$u(t) > u_1(t) \quad \textit{for}\ \ t \in [0, \omega],$$
$$\min\big\{u(t) - u_2(t) : \ t \in [0, \omega]\big\} < 0, \quad \max\big\{u(t) - u_2(t) : \ t \in [0, \omega]\big\} > 0.$$

(3) *If $\mu = \mu_0$, then problem (1) has a unique positive solution.*

(4) *If $\mu > \mu_0$, then problem (1) has no positive solution.*

Open question. The following question remains open in Theorem 1: What happens in the case of $\mu < 0$?

We now provide lower and upper estimates of the number μ_0 appearing in the conclusion of Theorem 1.

Proposition 2. *Let $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, h satisfy (7), and f be such that (4) holds. Then, the number μ_0 appearing in the conclusion of Theorem 1 satisfies*

$$\mu_0 \geq \frac{(\lambda-1)[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda\int\limits_0^\omega h(s)\,\mathrm{d}s\big]^{\frac{1}{\lambda-1}}\int\limits_0^\omega [f(s)]_-\,\mathrm{d}s}\,,$$

where Δ is a number depending on the coefficient p only, and

$$\mu_0 < \frac{(\lambda-1)\big[\mathrm{e}^{\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s - \int\limits_0^\omega [p(s)]_-\,\mathrm{d}s\big]^{\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda\int\limits_0^\omega h(s)\,\mathrm{d}s\big]^{\frac{1}{\lambda-1}}\big[\int\limits_0^\omega [f(s)]_-\,\mathrm{d}s - \mathrm{e}^{\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s}\int\limits_0^\omega [f(s)]_+\,\mathrm{d}s\big]}\,.$$

Remark 3. Let $\omega > 0$ and put $p(t) := a$, $h(t) := b$, $f(t) := -1$ for $t \in [0, \omega]$, where $a, b > 0$. Then, $p \in \mathcal{V}^-(\omega)$, h and f satisfy (7) and (5), respectively, and conclusions of Theorem 1 extend conclusions (2)–(4) of Proposition 1 for the non-autonomous Duffing equations with a sign-changing forcing term. Moreover, one can show that the number μ_0 appearing in Proposition 2 satisfies

$$\left(\frac{1}{\cosh\frac{\omega\sqrt{a}}{2}}\right)^{\frac{\lambda}{\lambda-1}} \frac{(\lambda-1)a}{\lambda} \left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}} < \mu_0 < \left(\mathrm{e}^{\frac{\omega^2 a}{4}}\right)^{\frac{\lambda}{\lambda-1}} \frac{(\lambda-1)a}{\lambda} \left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}};$$

compare it with the number appearing in Proposition 1.

If the forcing term f is non-positive, then Theorem 1 can be refined as follows.

Corollary. *Let $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, and conditions* (5), (7), *and* (8) *be satisfied. Then, there exists $\mu_0 \in\,]0, +\infty[$ such that the following conclusions hold:*

(1) *If $\mu = 0$, then problem* (1) *has a unique positive solution.*

(2) *If $0 < \mu < \mu_0$, then problem* (1) *has exactly two positive solutions u_1, u_2 and these solutions satisfy*

$$u_1(t) \neq u_2(t) \quad \textit{for } t \in [0, \omega].$$

(3) *If $\mu = \mu_0$, then problem* (1) *has a unique positive solution.*

(4) *If $\mu > \mu_0$, then problem* (1) *has no positive solution.*

Acknowledgements

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