Algorithm for Constructing Uniform Asymptotics of a Solution for Problem for Singular Perturbed Systems of Differential Equations with Differential Turning Point

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We consider the following system of differential equations with turning point (SSPDE):

$$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{0.1}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x),$$

is a known matrix,

$$\mathbf{A_0}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

when $\varepsilon \to 0$, $x \in [-l, l]$, $Y(x,\varepsilon) \equiv Y_k(x,\varepsilon) = \mathrm{colomn}(y_1(x,\varepsilon), y_2(x,\varepsilon), y_3(x,\varepsilon))$ is an unknown vector function, $H(x) = \mathrm{colomn}(0, 0, h(x))$ is a given vector function.

The scalar reduced equation for this matrix will be

$$x\widetilde{a}(x)\omega'(x) + b(x)\omega(x) = h(x).$$

The characteristic equation that corresponds to the SP system (0.1) is as follows:

$$A(x,0) - \lambda E| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ -b(x) & -a(x) & -\lambda \end{vmatrix} = -\lambda^3 - x\widetilde{a}(x)\lambda = 0.$$

The roots of this equation are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{x\widetilde{a}(x)}\,.$$

The purpose of this work is to construct uniform asymptotic of a solution for a given SSPDE with a stable turning point of the first kind.

1 Regularization of singularly perturbed systems of differential equations

In order to save all essential singular functions that appear in the solution of system (0.1) due to the special point $\varepsilon = 0$, a regularizing variable is introduced $t = \varepsilon^{-p} \cdot \varphi(x)$, where exponent p and regularizing function $\varphi(x)$ are to be determined.

Instead of $Y_k(x,\varepsilon)$ function $\widetilde{Y}_k(x,t,\varepsilon)$ transformation function will be studied, also the transformation will be performed in such a way that the following identity is true

$$\widetilde{Y}(x,t,\varepsilon)\Big|_{t=\varepsilon^{-p}\varphi(x)} \equiv Y(x,\varepsilon),$$

which is the necessary condition for a suggested method. The vector equation (0.1) can be written as

$$\widetilde{L}_\varepsilon \widetilde{Y}_k(x,t,\varepsilon) \equiv \mu\varphi' \, \frac{\partial \widetilde{Y}(x,t,\varepsilon)}{\partial t} + \mu^3 \, \frac{\partial \widetilde{y}(x,t,\varepsilon)}{\partial x} - A(x,\varepsilon)\widetilde{Y}_k(x,t,\varepsilon) = H(x). \tag{1.1}$$

Asymptotic forms of solutions for equation (1.1) are constructed in the form of the series

$$\widetilde{Y}_k(x,t,\varepsilon) = \sum_{i=1}^{2} D_i(x,t,\varepsilon) + f(x,\varepsilon)\nu(t) + \varepsilon^\gamma g(x,\varepsilon)\nu'(t) + \omega(x,\varepsilon),$$

$$\sum_{i=1}^{2} D_i(x,t,\varepsilon) = \begin{pmatrix} \varepsilon^{s1}\alpha_{k1}(x,\varepsilon) \\ \varepsilon^{s2}\alpha_{k2}(x,\varepsilon) \\ \varepsilon^{s3}\alpha_{k3}(x,\varepsilon) \end{pmatrix} U_i(t) + \varepsilon^\gamma \begin{pmatrix} \varepsilon^{k1}\beta_{k1}(x,\varepsilon) \\ \varepsilon^{k2}\beta_{k2}(x,\varepsilon) \\ \varepsilon^{k3}\beta_{k3}(x,\varepsilon) \end{pmatrix} {U_i}'(t),$$

where $U_1(t)$, $U_2(t)$ are the Airy–Langer functions [3] and $\alpha_{ik}(x,\varepsilon)$, $\beta_{ik}(x,\varepsilon)$, $f_k(x,\varepsilon)$, $g_k(x,\varepsilon)$, $\omega_k(x,\varepsilon)$, $k = \overline{1,3}$ are analytic functions with reference to a small parameter and are infinitely differentiable functions of variable $x \in [-l;l]$ which are still to be determined.

First of all, the analysis how transformation operator $\widetilde{L}_\varepsilon$ operates on vector function $D_k(x,t,\varepsilon)$ will be performed, and then the obtained result will be utilized in the homogeneous transformation equation (0.1). The following equation is obtained

$$\begin{aligned}
&\widetilde{L}_\varepsilon(\alpha_{ik}(x,\varepsilon)U_i(t) + \varepsilon^\gamma \beta_{ik}(x,\varepsilon)U_i'(t)) \\
&\quad = \varepsilon^{1-p}\alpha_{ik}(x,\varepsilon)\varphi'(x)U_i'(t) - \varepsilon^{1+\gamma-2p}\beta_{ik}(x,\varepsilon)\varphi'(x)\varphi(x)U_i(t) - A(x,\varepsilon)\alpha_k(x,\varepsilon)U_i(t) \\
&\qquad\qquad - \varepsilon^\gamma A(x,\varepsilon)\beta_{ik}(x,\varepsilon)U_i'(t) + \varepsilon\alpha_{ik}'(x)U_i(t) + \varepsilon^{1+\gamma}\beta_k'(x)U_i'(t) = 0.
\end{aligned}$$

Then, after equating corresponding coefficients of essential singular functions $U_k(t)$, $k = 1,2$ and their derivatives, two following vector equations are obtained:

$$U_i'(t):\ \varepsilon^{1-p}\alpha_{ik}(x,\varepsilon)\varphi'(x) - \varepsilon^\gamma\big[A_0(x) + \varepsilon A_1\big]\beta_{ik}(x,\varepsilon) = -\varepsilon^{1+\gamma}\beta_{ik}'(x,\varepsilon), \tag{1.2}$$

$$U_i(t):\ -\varepsilon^{1+\gamma-2p}\beta_{ik}(x,\varepsilon)\varphi(x)\varphi'(x) - \big[A_0(x) + \varepsilon A_1\big]\alpha_{ik}(x,\varepsilon) = -\varepsilon\alpha_{ik}'(x,\varepsilon). \tag{1.3}$$

2 Construction of formal solutions of a homogeneous transformation system

The unknown coefficients of the vector equations (1.2) and (1.3) are sought as following vector function series $(i = 1,2)$:

$$\alpha_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \alpha_{ikr}(x), \quad \beta_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \beta_{ikr}(x).$$

To determine vector function components $\alpha_{ikr} = \operatorname{colomn}(\alpha_{i1r}(x), \alpha_{i2r}(x), \alpha_{i3r}(x))$ and $\beta_{ikr}(x) = \operatorname{colomn}(\beta_{i1r}(x), \beta_{i2r}(x), \beta_{i3r}(x))$, the following recurrent systems of equations are obtained:

$$\begin{aligned}
&\Phi(x)Z_{k0}(x) = 0, \quad r = 0,1,2, \\
&\Phi(x)Z_{kr}(x) = FZ_{k(r-3)}(x), \quad r \geq 3.
\end{aligned} \tag{2.1}$$

At the moment, the regularizing function has not yet been defined; therefore, it will be defined as a solution of the problem

$$\varphi'^2\varphi(x) = x, \quad \varphi(0) = 0,$$

which is the following function

$$\varphi(x) = x.$$

The regularizing function of such kind has been considered in [3, 5].

Due to such a choice of the regularizing variable $\varphi(x)$, there is a nontrivial solution of the homogeneous system $\Phi(x)Z_{kr}(x) = 0$, $r = \overline{0,2}$, that is

$$Z_{ikr}(x) = \text{colomn}\left(0, \beta_{i3r}(x), -\beta_{i2r}(x), 0, \beta_{i2r}(x), \beta_{i3r}(x)\right),$$

where $\beta_{ksr}(x)$, $i = 1, 2$, $s = 2, 3$ are arbitrary up to some point and sufficiently smooth function at $x \in [0; l]$.

Solving systems of recurrent equations at the third step, i.e., when $r = 3$, and taking into account the already obtained solution (2.1), the following systems of algebraic equations in $\alpha_{kr}(x)$ and $\beta_{kr}(x)$ are obtained

$$\begin{cases} \alpha_{i13}(x) = \beta_{i20}(x) - \beta'_{i10}(x) \equiv \beta_{i20}(x), \\ \alpha_{i23}(x) - \beta_{i33}(x) = -\beta'_{i20}(x), \\ \alpha_{i33}(x) - \beta_{i13}(x) + \beta_{i23}(x) = -\beta'_{i30}(x), \end{cases} \tag{2.2}$$

and

$$\begin{cases} x\beta_{i13}(x) = -\alpha'_{i10}(x) + \alpha_{i20}(x) \equiv \alpha_{i20}(x) \equiv \beta_{i30}(x), \\ x\beta_{i23}(x) + \alpha_{i33}(x) = \alpha'_{i20}(x) \equiv (\beta_{i30}(x))', \\ x\beta_{i33}(x) + \alpha_{i13}(x) + \alpha_{i23}(x) = \alpha'_{i30}(x) \equiv [-x\beta_{i20}(x)]'. \end{cases} \tag{2.3}$$

Taking into account the fact that the functions are arbitrary, $\beta_{is0}(x) = \beta^0_{is0} \cdot \hat{\beta}_{is0}(x)$, $i = 1, 2$, $s = 2, 3$, where $\beta^0_{is0}(x)$ are an arbitrary constants, $\hat{\beta}_{is0}(x)$ is a partial and sufficiently smooth for all $x \in [-l; l]$ solutions of homogeneous equations. This definition of vector functions $Z_{ik0}(x)$ implies that there are following solutions of inhomogeneous systems of algebraic equations (2.2) and (2.3):

$$Z_{ik3}(x) = \text{colomn}\left(z_{i13}, z_{i23}, z_{i33}, z_{i43}, z_{i53}, z_{i63}\right),$$

$$z_{i13} = \beta_{i20}(x), \quad z_{i23} = -\beta'_{i20}(x) + \beta_{i33}(x), \quad z_{i33} = -\beta'_{i30}(x) - \beta_{i23}(x) + \frac{\beta_{i30}}{x}, \quad z_{i43} = \frac{\beta_{i20}(x)}{x},$$

$$z_{i53} = \beta_{i21}(x), \quad z_{i63} = \beta_{i31}(x),$$

where $\beta_{i21}(x)$ and $\beta_{i31}(x)$ are arbitrary up to some point and sufficiently smooth functions for all $x \in [-l; l]$.

Thus, gradual solving of systems of equations (2.2) and (2.3) gives two formal solutions of the transformation vector equation (0.1)

$$D_{ik}(x, \varepsilon^{-\frac{2}{3}}\varphi(x), \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \left[\alpha_{ikr}(x)U_i(\varepsilon^{-\frac{2}{3}}\varphi(x)) + \varepsilon^{\frac{1}{3}}\beta_{ikr}(x, \varepsilon)U'_i(\varepsilon^{-\frac{2}{3}}\varphi(x))\right].$$

The third formal solution of the homogeneous vector equation (0.1) is then constructed as a series

$$\omega(x, \varepsilon) \equiv \sum_{r=0}^{\infty} \varepsilon^r \omega_r(x) \equiv \text{colon}\left(\sum_{r=0}^{\infty} \varepsilon^r \omega_{1r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{2r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{3r}(x)\right). \tag{2.4}$$

Substituting solution (2.4) into equation (0.1), the following recurrent system of differential equations can be obtained:

$$\begin{aligned}
&A_0(x)\omega_0(x) = 0,\\
&A_r(x)\omega_r(x) = -A_1(x)\omega_{(r-1)}(x) - \omega'_{(r-1)}(x), \ \ r \geq 1.
\end{aligned}$$

Then, solving these systems step by step, the following zero approximation can be constructed

$$\omega_0(x) = \mathrm{colomn}(\omega_{10}(x), \omega_{20}(x), \omega_{30}(x)) \equiv \mathrm{colomn}(\omega^0_{10} \cdot x, -\omega^0_{10}, 0),$$

that has only one arbitrary constant ω^0_{01}.

3 Construction of formal partial solutions

Similarly to the previous steps, in order to construct asymptotic forms of partial solutions of the inhomogeneous transformation vector equation (0.1), let us analyze how transformation operator operates on an element from the space of non-resonant solutions

$$f(x,\varepsilon)\psi(t) + \varepsilon^\gamma g(x,\varepsilon)\psi'(t) + \overline{\omega}(x,\varepsilon).$$

Consequently, the following systems are obtained

$$\psi'(t): \ f_k(x,\varepsilon) - \big[A_0(x) + \mu^3 A_1\big] g_k(x,\varepsilon) = -\mu^3 g'_k(x,\varepsilon), \tag{3.1}$$

$$\psi(t): \ x g_k(x,\varepsilon) + \big[A_0(x) + \mu^3 A_1\big] f_k(x,\varepsilon) = \mu^3 f'_k(x,\varepsilon), \tag{3.2}$$

$$\mu^3 \overline{\omega}'(x,\varepsilon) - \big[A_0(x) + \mu^3 A_1\big]\overline{\omega}(x,\varepsilon) + \mu^2 g_k(x,\varepsilon) = H(x). \tag{3.3}$$

In order to have smooth solutions of systems (3.1)–(3.3), the asymptotic forms of the solutions are constructed as series

$$f_k(x,\varepsilon) = \sum_{r=-2}^{+\infty} \mu^r f_r(x), \quad g_k(x,\varepsilon) = \sum_{r=-2}^{+\infty} \mu^r g_r(x), \quad \overline{\omega}(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \overline{\omega}_r(x).$$

To determine components of the vector functions $f_{kr} = \mathrm{colomn}(f_{1r}(x), f_{2r}(x), f_{3r}(x))$ and $g_{kr}(x) = \mathrm{colomn}(g_{1r}(x), g_{2r}(x), g_{3r}(x))$, the following recurrent systems of equations are obtained:

$$\begin{aligned}
&\Phi(x) Z^{part.}_{k0}(x) = 0, \ \ r = -2, -1, 0,\\
&\Phi(x) Z^{part.}_{kr}(x) = -Z^{part.}_{k(r-3)}(x), \ \ r \geq 1.
\end{aligned}$$

Then, to determine the vector functions $\overline{\omega}_r(x)$, the following recurrent systems of equations are obtained as well

$$\begin{aligned}
&-A_0(x)\overline{\omega}_{kr}(x) = H(x) - g_{k(r-2)}(x), \ \ r = 0,\\
&-A_0(x)\overline{\omega}_{kr}(x) = -g_{k(r-2)}(x), \ \ r = 1, 2,\\
&\overline{\omega}'_{k(r-3)}(x) - A_0(x)\overline{\omega}_{kr}(x) = -g_{k(r-3)}(x) + A_1\overline{\omega}_{k(r-3)}(x), \ \ r \geq 3,
\end{aligned}$$

where $\overline{\omega}_r(x) = \mathrm{colomn}(\overline{\omega}_{1r}(x), \overline{\omega}_{2r}(x), \overline{\omega}_{3r}(x))$ is an unknown vector function. Doing further iterations, functions $\overline{\omega}_r(x)$, $f_r(x)$, $g_r(x)$, which are sufficiently smooth in the whole domain, are obtained. Therefore, the partial solution of the transformation vector equation (0.1) is then defined as the series

$$\widetilde{Y}^{part.}_k(x,t,\varepsilon) = \sum_{r=-2}^{\infty} \mu^r \big[f_{kr}(x)\nu(t) + \mu g_{kr}(x)\nu'(t)\big] + \sum_{r=0}^{\infty} \mu^r \overline{\omega}_{kr}(x).$$

4 Conclusions

Thus, the transformation vector equation (0.1) has three formal solutions in form of the series

$$\widetilde{Y}(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^{r}\Big[\sum_{i=1}^{2}\Big[\alpha_{ikr}(x)U_i(\varepsilon^{-\frac{2}{3}}\cdot x)+\varepsilon^{\frac{1}{3}}\beta_{kr}(x)\,\frac{dU_i(\varepsilon^{\frac{2}{3}}\cdot x)}{d(\varepsilon^{-\frac{2}{3}}\cdot x)}\Big]\Big]$$
$$+\sum_{r=-2}^{\infty}\varepsilon^{r}\Big[f_{kr}(x)\nu(\varepsilon^{\frac{2}{3}}\cdot x)+\varepsilon^{\frac{1}{3}}g_{kr}(x)\,\frac{d\nu(\varepsilon^{-\frac{2}{3}}\cdot x)}{d(\varepsilon^{-\frac{2}{3}}\cdot x)}\Big]+\sum_{r=0}^{\infty}\varepsilon^{r}\overline{\omega}_{kr}(x).$$

5 Algorithm for constructing the asymptotics of a solution of the system

Let us write the main result of this paper in the following algorithm:

Step I. An extension of the singularly perturbed problem. In a singularly perturbed system with a turning point next to an independent one variable x introduces a new vector-variable $t=\varepsilon^{-p}\cdot\varphi(x)$. Then instead of the wanted one vector-function $Y(x,\varepsilon)$ a new "extended vector-function" $\widetilde{Y}(x,t,\varepsilon)$ is studied. The expansion is carried out in such a way that the condition as in regularization method

$$\widetilde{Y}(x,t,\varepsilon)\Big|_{t=\varepsilon^{-p}\cdot\varphi(x)}\equiv Y(x,\varepsilon).$$

p and $\varphi(x)$ are determined for each specific case. There is a transition from a problem with one variable to a problem with two variables t and x.

Step II. The space of resonance-free solutions. For regularization, a specific space of functions is introduced, this space is called *the space of resonance-free solutions* and for each specific problem this space has its own specificity

$$\sum_{k=1}^{2}D_k(x,t,\varepsilon),f_k(x,\varepsilon)\psi(t),\varepsilon^{\gamma}g_k(x,\varepsilon)\psi'(t),\omega_k(x,\varepsilon).$$

Step III. Regularization of a singularly perturbed problem. The extended problem is studied in the space of resonance-free solutions and is reduced to an equation in which the small parameter $\varepsilon>0$ enters regularly.

Step IV. The formalism of constructing a solution to the problem. Since the extended problem is regularly perturbed with respect to the small one parameter in the space of resonance-free solutions, then we will look for the solution of the problem in the form of a series

$$\widetilde{Y}(x,t,\mu)=\sum_{r=-2}^{\infty}\mu^{r}Y(x), \tag{5.1}$$

where $\mu=\sqrt[3]{\varepsilon}$ is a small parameter.

We start the construction of the asymptotic series with negative powers of a small parameter in order to obtain uniform asymptotics intersection of the SSPDE. The right part of the system will have a break of the second kind at the turning point. Therefore, in general, it will not belong set of values of the main extended operator $\widetilde{L}_{\varepsilon}$. By substituting series (5.1) in system (1.1), to determine the coefficients of this series, we will get some system of pointwise recurrent equations with initial or boundary conditions.

Step V. Construction of formal solutions of homogeneous extended system. Those obtained in the previous point are recurrent the equation for determining the coefficients of series (5.1) is partial differential equations with point boundary conditions. We will show that this system of equations is asymptotically correct in the space of resonance-free solutions D_k. At this stage, the theory of existence is developed of the iterative equation of the form

$$\Phi(x)\cdot Z_{kr}(x)=F\cdot Z_{kr}(x),$$

where $\Phi(x)$ is the matrix of system (1.1), $Z_{kr}(x)$ is a column vector composed of analytic functions $\theta_1(x,\varepsilon)$. And the first members are being built of the asymptotic solution of the homogeneous problem under consideration.

Step VI. Construction of formal inhomogeneous solutions extended system. In this section, a function is being built for the inhomogeneous problem using a recurrent equation

$$\Phi(x)\cdot Z_{kr}(x)=F\cdot Z_{kr}(x),$$

where $\Phi(x)$ is the matrix of system (1.1), $Z_{kr}(x)$ is a column vector composed of analytic functions $\theta_2(x,\varepsilon)$.

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