Definition of Total Wandering and Total Nonwandering of a Differential System and their Study at the First Approximation

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For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider a nonlinear, generally speaking, differential system of the form

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \quad t \in \mathbb{R}_+ \equiv [0,+\infty), \quad x \in G, \tag{1}$$

where the right-hand side satisfies the condition $f, f'_x \in C(\mathbb{R}_+ \times G)$ and the zero solution is allowed.

We associate with system (1) the linear homogeneous system of its *first approximation*

$$\dot{x} = A(t)x \equiv f_{\prime}(t,x), \quad A(t) \equiv f'_x(t,0), \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n, \tag{2}$$

for which we do not require here the uniformity in $t \in \mathbb{R}_+$ of the natural (pointwise) smallness of the nonlinear addition

$$h(t,x) \equiv f(t,x) - A(t)x = o(x), \quad x \to 0.$$

Denote by $x_f(\,\cdot\,,x_0)$ and $S_*(f)$ or $S_\delta(f)$ a *non-extendable* solution of system (1) with the initial condition $x_f(0,x_0) = x_0$ and sets of solutions with initial values x_0, satisfying the conditions $|x_0| \neq 0$ or, respectively, $0 < |x_0| < \delta$.

Definition 1. *Wandering functional* $\mathrm{P}(u,t)$, defined for numbers $t \in \mathbb{R}_+$ and continuously-differentiable functions $u : [0,t] \to \mathbb{R}^n \setminus \{0\}$, is given by the formula

$$\mathrm{P}(t,u) \equiv \int\limits_0^t \left| \Big(\frac{u(\tau)}{|u(\tau)|} \Big)^{\cdot} \right| d\tau, \quad \tau \in [0,t],$$

adding that whenever the function u is not defined on the entire segment $[0,t]$, it takes the value $+\infty$. For each system (1), momentum $t \in \mathbb{R}_+$, and non-degenerate transformation $L \in \operatorname{Aut}\mathbb{R}^n$ we define the values of the *lower* and the *upper ball wandering functionals*, given respectively by the equalities

$$\check{\mathrm{P}}_b(f,t,L) \equiv \varliminf_{x_0 \to 0} \mathrm{P}(t, Lx_f(\,\cdot\,,x_0)), \quad \hat{\mathrm{P}}_b(f,t,L) \equiv \varlimsup_{x_0 \to 0} \mathrm{P}(t, Lx_f(\,\cdot\,,x_0)). \tag{3}$$

Lower weak $\check{\rho}_b^\circ(f)$ and *strong* $\check{\rho}_b^\bullet(f)$ *ball wandering indicators* of system (1) are given by the formulas

$$\check{\rho}_b^\circ(f) \equiv \varliminf_{t \to +\infty} \inf_{L \in \operatorname{Aut}\mathbb{R}^n} t^{-1}\check{\mathrm{P}}_b(f,t,L), \quad \check{\rho}_b^\bullet(f) \equiv \inf_{L \in \operatorname{Aut}\mathbb{R}^n} \varliminf_{t \to +\infty} t^{-1}\check{\mathrm{P}}_b(f,t,L), \tag{4}$$

and *upper weak* $\hat{\rho}_b^\circ(f)$ and *strong* $\hat{\rho}_b^\bullet(f)$ *ball wandering indicators* – by the same formulas (4) respectively, but with the upper limits at $t \to +\infty$ instead of the lower ones

$$\hat{\rho}_b^\circ(f) \equiv \varlimsup_{t \to +\infty} \inf_{L \in \operatorname{Aut}\mathbb{R}^n} t^{-1}\hat{\mathrm{P}}_b(f,t,L), \quad \hat{\rho}_b^\bullet(f) \equiv \inf_{L \in \operatorname{Aut}\mathbb{R}^n} \varlimsup_{t \to +\infty} t^{-1}\hat{\mathrm{P}}_b(f,t,L). \tag{5}$$

The indicators $\check{\rho}_b^{\circ}(f)$ and $\hat{\rho}_b^{\bullet}(f)$ turn out to be respectively the smallest and the largest of four ball wandering indicators (4), (5) of system (1) introduced in Definition 1.

Other functionals are also known that are responsible for similar properties of solutions not related to their norm (see, for example, [1–3]): the *oscillation* or the *oriented, non-oriented, frequency* and *flat rotation*, as well as the *rotation* of the given *rank*. In addition to the ball indicators, we can also consider the *spherical* or the *radial* ones [4].

The total wandering of a differential system defined below (near its zero solution, which we will not mention further for brevity) remotely resembles Lyapunov stability. In contrast to stability, wandering does not mean that all solutions that start close enough to zero remain forever in its given neighborhood, but that their average (in time) angular velocity is positive and even separated from zero (uniformly in all these solutions at once). However, in the nonlinear case, the matter is complicated by the fact that the solutions mentioned may not be defined on the entire time semiaxis. The situation is similar with complete nonwandering.

Definition 2. We say that system (1) has:

1) *complete wandering* if there exist $\varepsilon > 0$ and $T \in \mathbb{R}_+$ such that for each $L \in \operatorname{Aut}\mathbb{R}^n$ and $t > T$ the estimate holds
$$\check{\mathrm{P}}(f,t,L) > \varepsilon t;$$
2) *complete nonwandering* if for any $\varepsilon > 0$ there exist $T \in \mathbb{R}_+$ and $L \in \operatorname{Aut}\mathbb{R}^n$, that for every $t > T$ the estimate holds
$$\hat{\mathrm{P}}(f,t,L) < \varepsilon t.$$

Whether a system is completely wandering or nonwandering is uniquely determined by the signs of its corresponding ball wandering indicators.

Theorem 1. *The complete wandering and the complete nonwandering of system* (1) *are equivalent to the positiveness of its lower ball wandering indicator*
$$\check{\rho}_b^{\circ}(f) > 0$$
and, respectively, to the equality to zero of its upper ball wandering indicator
$$\hat{\rho}_b^{\bullet}(f) = 0.$$

All the ball wandering indicators of a system coincide with the corresponding indicators of the system of its first approximation (which are calculated much easier, since in the case of a linear system in formulas (3) the lower and upper limits at $x_0 \to 0$ can be replaced by the exact lower and upper bounds over all $x_0 \neq 0$, respectively).

Theorem 2. *For any system* (1) *and system* (2) *of its first approximation, the equalities hold*
$$\tilde{\rho}_b^{*}(f) = \tilde{\rho}_b^{*}(f_\prime), \quad \tilde{} = \check{}\,,\hat{}\,, \quad * = \circ, \bullet.$$

Thus, both the complete wandering and the complete nonwandering of a nonlinear system are uniquely determined by the system of its first approximation.

Theorem 3. *The complete wandering and the complete nonwandering of system* (1) *are equivalent to the positiveness of the indicator of system* (2) *of its first approximation*
$$\check{\rho}_b^{\circ}(f_\prime) > 0$$
and, accordingly, to the equality to zero of the indicator of system (2)
$$\hat{\rho}_b^{\bullet}(f_\prime) = 0.$$

Definition 3 ([4])**.** For a system (1) and for its nonzero solution $x \in S_*(f)$ defined on the whole semiaxis $\mathbb{R}_+$, we define:

(a) *lower weak* and *strong wandering indicators* – by the formulas

$$\check{\rho}^\circ(x) \equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\mathrm{P}(t,Lx), \quad \check{\rho}^\bullet(x) \equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\mathrm{P}(t,Lx); \tag{6}$$

(b) *upper weak* and *strong wandering indicators* – by the same formulas (6) respectively, but with the upper limits at $t \to +\infty$ instead of the lower ones

$$\hat{\rho}^\circ(x) \equiv \varlimsup_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\mathrm{P}(t,Lx), \quad \hat{\rho}^\bullet(x) \equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varlimsup_{t\to+\infty} t^{-1}\mathrm{P}(t,Lx); \tag{7}$$

(c) *exact* or *absolute* varieties of indicators (4)–(7) that arise when the corresponding values of the lower and upper indicators or, respectively, weak and strong ones coincide: in the first case, we will omit the checkmark and the cap in their designation, and in the second one – an empty and full circle.

Surprisingly, the presence of a complete wandering system does not mean that it has at least one solution with a positive wandering indicator, and vice versa, the presence of complete nonwandering system does not mean that it has at least one solution with a zero wandering indicator.

Theorem 4. *For $n = 2$, there exist two Lyapunov stable systems* (1)*, which, like all their nonzero solutions defined on the entire semiaxis $\mathbb{R}_+$, have exact absolute wandering indicators: one of these systems has complete wandering, is periodic, and satisfies the conditions*

$$\rho_b(f) = 1 > 0 = \rho(x), \quad x \in S_*(f),$$

while the other system has complete nonwandering and satisfies the conditions

$$\rho_b(f) = 0 < 1 = \rho(x), \quad x \in S_*(f).$$

References

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