Necessary Solvability Conditions for Non-Linear Integral Boundary Value Problems

A. Rontó

Brno University of Technology, Faculty of Business and Management
Brno, Czech Republic
E-mail: rontoa@fbm.vutbr.cz

M. Rontó

Institute of Mathematics, University of Miskolc, Miskolc, Hungary
E-mail: matronto@uni-miskolc.hu

I. Varga

Faculty of Mathematics, Uzhhorod National University, Uzhhorod, Ukraine
E-mail: iana.varga@uzhnu.edu.ua

We study the following non-linear integral boundary value problem

$$\frac{dx(t)}{dt} = f(t, x(t)), \ \ t \in [a, b], \quad \int\limits_a^b g(s, x(s))\, ds = d, \tag{1}$$

where $f \in C([a, b] \times D; \mathbb{R}^n)$, $g \in C([a, b] \times D; \mathbb{R}^n)$, $d \in \mathbb{R}^n$ is a given vector and the domain $D \subset \mathbb{R}^n$ will be specified later (See, (7), (8)). Moreover, we suppose that $f \in Lip(K, D)$, $g \in Lip(K_g, D)$, i.e., f and g locally Lipsichitzian

$$\begin{gathered} |f(t, u) - f(t, v)| \le K|u - v|, \ \text{ for all } \ \{u, v\} \subset D \ \text{ and } \ t \in [a, b], \\ |g(t, u) - g(t, v)| \le K_g|u - v|, \ \text{ for all } \ \{u, v\} \subset D \ \text{ and } \ t \in [a, b]. \end{gathered} \tag{2}$$

To study the BVP (1) we will use an approach similar to that of [1].

For vectors $x = col(x_1, \dots, x_n) \in \mathbf{R}^n$ the notation $|x| = col(|x_1|, \dots, |x_n|)$ is used and the inequalities between vectors are understood componentwise. The same convention is adopted for operations like "max" and "min". For any non-negative vector $\rho \in \mathbf{R}^n$ under the componentwise ρ-neighbourhood of a point $z \in \mathbf{R}^n$ we understand the set

$$O_\rho(z) := \big\{\xi \in \mathbf{R}^n : \ \ |\xi - z| \le \rho\big\}. \tag{3}$$

Similarly, the ρ-neighbourhood of a domain $\Omega \subset \mathbf{R}^n$ is defined as

$$O_\rho(\Omega) := \bigcup_{z \in \Omega} O_\rho(z). \tag{4}$$

A particular kind of vector ρ will be specified below in relations (7), (8).

I_n is the identity matrix of dimension n. $r(K)$ is the maximal, in modulus, eigenvalue of the matrix K. We also assume that

$$r(Q) < 1, \ \ Q = \frac{3(b - a)}{10}\, K. \tag{5}$$

Let us choose certain compact convex sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, and define the set

$$D_{a,b} := (1-\theta)z + \theta\eta, \quad z \in D_a, \quad \eta \in D_b, \quad \theta \in [0,1] \tag{6}$$

and according to (4) its ρ-neighbourhood

$$D = O_\rho(D_{a,b}) \tag{7}$$

with a non-negative vector $\rho = col(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ such that

$$\rho \geq \frac{b-a}{2}\, \delta_{[a,b],D}(f), \tag{8}$$

where $\delta_{[a,b],D}(f)$ denotes the 1/2 of oscillation of function f over $[a,b] \times D \times D$

$$\delta_{[a,b],D}(f) := \frac{\max\limits_{(t,x)\in[a,b]\times D} f(t,x) - \min\limits_{(t,x)\in[a,b]\times D} f(t,x)}{2}. \tag{9}$$

Instead of the original boundary value problem (1) we will consider the family of auxiliary two-point parametrized boundary value problems

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \in [a,b], \tag{10}$$

$$x(a) = z, \quad x(b) = \eta, \tag{11}$$

where z and η are treated as free parameters.

Let us connect with problem (10), (11) the sequence of functions

$$x_{m+1}(t,z,\eta) = z + \int_a^t f(s, x_m(s,z,\eta))\, ds$$

$$- \frac{t-a}{b-a} \int_a^b f(s, x_m(s,z,\eta))\, ds + \frac{t-a}{b-a}[\eta - z], \quad t \in [a,b], \quad m = 0,1,2,\ldots, \tag{12}$$

satisfying (11) for arbitrary $z, \eta \in \mathbb{R}^n$, where

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}[\eta - z] = \Big(1 - \frac{t-a}{b-a}\Big)z + \frac{t-a}{b-a}\eta, \quad t \in [a,b]. \tag{13}$$

It is easy to see from (13) that $x_0(t,z,\eta)$ is a linear combination of vectors z and η, when $z \in D_a$, $\eta \in D_b$.

We have previously proved the following statements.

Theorem 1 (Uniform convergence)**.** *Let conditions* (2), (5), (8) *be fulfilled.*
Then, for all fixed $(z,\eta) \in D_a \times D_b$ *we have*

1. *The functions of sequence* (12) *belonging to the domain* D *of form* (7) *are continuously differentiable on the interval* $[a,b]$ *and satisfy conditions* (11).

2. *The sequence of functions* (12) *for* $t \in [a,b]$ *converges uniformly as* $m \to \infty$ *with respect to the domain* $(t,z,\eta) \in [a,b] \times D_a \times D_b$ *to the limit function*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta), \tag{14}$$

satisfying conditions (11).

3. *The function* $x_\infty(t,z,\eta)$ *for all* $t\in[a,b]$ *is a unique continuously differentiable solution of the integral equation*

$$x(t)=z+\int_a^t f(s,x(s))\,ds-\frac{t-a}{b-a}\int_a^b f(s,x(s))\,ds+\frac{t-a}{b-a}\,[\eta-z], \tag{15}$$

i.e., it is the solution to the Cauchy problem for the modified system of integro-differential equations

$$\frac{dx}{dt}=f(t,x(t))+\frac{1}{b-a}\,\Delta(z,\eta),\ \ x(a)=z, \tag{16}$$

where $\Delta(z,\eta):D_a\times D_b\to\mathbb{R}^n$ *is a mapping given by the formula*

$$\Delta(z,\eta)=[\eta-z]-\int_a^b f(s,x_\infty(s,z,\eta))\,ds. \tag{17}$$

4. *The error estimation*

$$\big|x_\infty(t,z,\eta)-x_m(t,z,\eta)\big|\leqslant\frac{10}{9}\,\alpha_1(t)Q^m(1_n-Q)^{-1}\delta_{[a,b],D}(f),\ \ t\in[a,b],\ \ m\ge 0 \tag{18}$$

holds, where

$$\alpha_1(t)=2(t-a)\Big(1-\frac{t-a}{b-a}\Big)\le\frac{b-a}{2}\,,\ \ t\in[a,b].$$

Theorem 2 (Relation $x_\infty(t,z,\eta)$ to the solution of the original boundary value problem (1))**.** *Under the assumptions of Theorem 1, the limit function* $x_\infty(t,z,\eta)=\lim\limits_{m\to\infty}x_m(t,z,\eta)$ *of sequence* (12) *is a solution to the integral boundary value problem* (1) *if and only if the pair of vector-parameters* (z,η) *satisfies the system of* $2n$ *determining algebraic equations*

$$\Delta(z,\eta):=[\eta-z]-\int_a^b f(s,x_\infty(s,z,\eta))\,ds=0,\quad \Lambda(z,\eta)=\int_a^t g(s,x_\infty(s,z,\eta))\,ds=d. \tag{19}$$

On the base of m*th approximate determining equations*

$$\Delta_m(z,\eta):=[\eta-z]-\int_a^b f(s,x_m(s,z,\eta))\,ds=0,\quad \Lambda_m(z,\eta)=\int_a^t g(s,x_m(s,z,\eta))\,ds=d \tag{20}$$

introduce the mapping $H_m:D_a\times D_b\to\mathbb{R}^{2n}$

$$H_m(z,\eta)=\begin{bmatrix}[\eta-z]-\displaystyle\int_a^b f(s,x_\infty(s,z,\eta))\,ds\\ \Lambda_m(z,\eta)-d\end{bmatrix}. \tag{21}$$

Theorem 3 (Sufficient conditions for the solvability of the integral boundary value problem (1))**.** *Assume that the conditions of Theorem 1 hold. Moreover, one can specify an* $m\ge 1$ *and set*

$\Omega \subset \mathbb{R}^{2n}$ of the form $\Omega := D_1 \times D_2$, where $D_1 \sqsubseteq D_a$, $D_2 \sqsubseteq D_b$ are certain bounded open sets, such that the mapping H_m, satisfies the relation

$$|H_m(z,\eta)| \rhd_{\partial\Omega} \begin{bmatrix} \dfrac{10(b-a)^2}{27}\, KQ^m(I_n-Q)^{-1}\delta_{[a,b],D}(f) \\ \dfrac{5(b-a)}{9}\, K_g Q^m(I_n-Q)^{-1}\delta_{[a,b],D}(f) \end{bmatrix} \tag{22}$$

on the boundary $\partial\Omega$, where the binary relation $\rhd_{\partial\Omega}$ in (22) means that for all $(z,\eta) \in \partial\Omega$ at least one of the components $k(z,\eta)$ of the vector $H_m(z,\eta)$ is greater than the corresponding component of the right hand side vector in (22). (One can see, that the number $k(z,\eta)$ of components depends on the point $(z,\eta) \in \partial\Omega$.)

If, in addition, the Brouwer's degree of the mapping H_m does not equal to zero, i.e.,

$$\deg(H_m, \Omega, 0) \neq 0, \tag{23}$$

then there exists a pair (z^*, η^*) from $D_1 \times D_2$ for which the function $x^*(\cdot) = x_\infty(\cdot, z^*, \eta^*)$ is a continuously differentiable solution to the boundary value problem (1), where $x_\infty(t, z^*, \eta^*) = \lim_{m\to\infty} x_m(t, z^*, \eta^*)$, $t \in [a,b]$.

In order to verify condition (22) of Theorem 3 one has to use the recurrence formula (12) to compute the function $x_m(\cdot, z, \eta)$ analytically, depending on the parameters z and η, at every point $(z,\eta) \in \partial\Omega$, verify whether at least one component of the $2n$-dimensional vector $|H_m(z,\eta)|$ is strictly greater than the corresponding component of the vector at right hand side of (22). Verification of the validity of (23) is a rather difficult problem in general. But in the smooth case, it follows directly from the definition of the topological degree, that if the Jacobian matrix of the function H_m in (21) is non-singular at its isolated zero (z_m^0, η_m^0), i.e.,

$$\det \frac{\partial}{\partial(z,\eta)}\, H_m(z_m^0, \eta_m^0) \neq 0,$$

then inequality (23) holds. The symbol $\frac{\partial}{\partial(z,\eta)}$ means the derivative with respect to the vector of variables $(z_1, \ldots, z_n, \eta_1, \ldots, \eta_n)$.

We proved the following lemma about the continuous dependence of the limit function $x_\infty(\cdot, z, \eta)$ and determining functions $\Delta(z,\eta), \Lambda(z,\eta)$ defined in (19) with respect to parameters $(z,\eta) \in D_a \times D_b$.

Lemma 1. *Let the assumptions of Theorem 1 be satisfied for the integral boundary value problem (1). Then for arbitrary pairs of parameters $(z', \eta') \in D_a \times D_b$ and $(z'', \eta'') \in D_a \times D_b$, the limit functions $x'_\infty(\cdot, z', \eta')$, $x''_\infty(\cdot, z'', \eta'')$ of sequence (12) for $t \in [a,b]$ satisfy the following Lipschitz-type condition*

$$\left|x'_\infty(\cdot, z', \eta') - x''_\infty(\cdot, z'', \eta'')\right| \leq \left[I_n + \frac{10}{9}\,\alpha_1(\cdot) K (I_n-Q)^{-1}\right]\left[\,|z'-z''| + |\eta'-\eta''|\,\right]. \tag{24}$$

Formulas (19) determine well defined functions $\Delta(z,\eta) : \mathbf{R}^{2n} \to \mathbf{R}^n$ and $\Lambda(z,\eta) : \mathbf{R}^{2n} \to \mathbf{R}^n$, which in addition satisfy the following Lipschitz-type estimates

$$\left|\Delta(z', \eta') - \Delta(z'', \eta'')\right| \leq \left[I_n + \Big((b-a)K + \frac{10}{27}\,(b-a)^2 K (I_n-Q)^{-1}\Big)\right]\left[\,|z'-z''| + |\eta'-\eta''|\,\right],$$

$$\left|\Lambda(z', \eta') - \Lambda(z'', \eta'')\right| \leq \left[\Big((b-a)K_g + \frac{10}{27}\,K_g (b-a)^2 K (I_n-Q)^{-1}\Big)\right]\left[\,|z'-z''| + |\eta'-\eta''|\,\right].$$

The following statement gives a condition which is necessary for the domain

$$\Omega = G_a \times G_b, \ \ G_a \sqsubseteq D_a, \ \ G_b \sqsubseteq D_b \tag{25}$$

to contain a pair of parameters (z^*, η^*) determining the solution

$$x(\,\cdot\,) = x_\infty(\,\cdot\,, z^*, \eta^*) = \lim_{m\to\infty} x_m(\,\cdot\,, z^*, \eta^*)$$

of the given integral boundary value problem (1).

Theorem 4. *Let the assumptions of Theorem 1 be satisfied for the integral boundary value problem* (1)*. Then for domain* (25) *to contain a pair of parameters* (z^*, η^*) *determining the solution* $x(\,\cdot\,)$ *of the given integral boundary value problem at the points* $t = a$ *and* $t = b$

$$x(a) = z^* \ \ \text{and} \ \ x(b) = \eta^*,$$

it is necessary that for all m *and arbitrary* $\widetilde{z} \in G_a$, $\widetilde{\eta} \in G_b$ *to be true for the approximate determining functions the following inequalities*

$$\begin{aligned}
\Delta_m(\widetilde{z}, \widetilde{\eta}) \le \sup_{z\in G_a,\, \eta\in G_b} \bigg[I_n + \Big((b-a)K + \frac{10}{27}\,(b-a)^2 K(I_n - Q)^{-1}\Big)\bigg] \big[\,|z' - z''| + |\eta' - \eta''|\,\big] \\
+ \frac{10}{27}\,(b-a)^2 K Q^m (1_n - Q)^{-1} \delta_{[a,b],D}(f), \\
\Lambda_m(\widetilde{z}, \widetilde{\eta}) \le \sup_{z\in G_a,\, \eta\in G_b} \bigg[\Big((b-a)K_g + \frac{10}{27}\,K_g(b-a)^2 K(I_n - Q)^{-1}\Big)\bigg] \big[\,|z' - z''| + |\eta' - \eta''|\,\big] \\
+ \frac{10}{27}\,(b-a)^2 K_g Q^m (1_n - Q)^{-1} \delta_{[a,b],D}(f).
\end{aligned}$$

References

[1] A. Rontó, M. Rontó and J. Varha, A new approach to non-local boundary value problems for ordinary differential systems. *Appl. Math. Comput.* **250** (2015), 689–700.