Non-Autonomous First Integrals of Autonomous Polynomial Hamiltonian Systems

A. Pranevich, A. Grin, E. Musafirov

Yanka Kupala State University of Grodno, Grodno, Belarus

E-mails: pranevich@grsu.by; grin@grsu.by; musafirov@bk.ru

F. Munteanu, C. Sterbeti

University of Craiova, Craiova, Romania

E-mails: munteanufm@gmail.com; sterbetiro@yahoo.com

1 Introduction

We consider a canonical Hamiltonian ordinary differential system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(q,p), \quad \frac{dp_i}{dt} = -\partial_{q_i} H(q,p), \ \ i = 1,\dots,n, \tag{1.1}$$

where $q = (q_1,\dots,q_n) \in \mathbb{R}^n$ and $p = (p_1,\dots,p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, $t \in \mathbb{R}$, and the Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a polynomial of degree $h \geqslant 2$.

In this paper, using the Darboux theory of integrability [3, 4] and the notion of partial integral (multiple partial integral, conditional partial integral) [5, 8–11], we study the existence of additional non-autonomous first integrals of the autonomous polynomial Hamiltonian system (1.1).

The Darboux theory of integrability (or the theory of partial integrals) was established by the French mathematician *Jean-Gaston Darboux* [3] in 1878, which provided a link between the existence of first integrals and invariant algebraic curves (or partial integrals) for polynomial autonomous differential systems. For the polynomial differential systems, the Darboux theory of integrability is one of the best theories for studying the existence of first integrals (see [4, 6, 12]).

To avoid ambiguity, we give the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(G)$ on a domain $G \subset \mathbb{R}^{2n}$ is the function

$$\big[u(q,p), v(q,p)\big] = \sum_{i=1}^{n} \big(\partial_{q_i} u(q,p)\, \partial_{p_i} v(q,p) - \partial_{p_i} u(q,p)\, \partial_{q_i} v(q,p)\big) \ \text{ for all } \ (q,p) \in G.$$

We say that [4, p. 20] the linear differential operator of first order

$$\mathfrak{B}(t,q,p) = \partial_t + \sum_{i=1}^{n} \Big(\partial_{p_i} H(q,p)\, \partial_{q_i} - \partial_{q_i} H(q,p)\, \partial_{p_i}\Big) \ \text{ for all } \ (t,q,p) \in \mathbb{R}^{2n+1}$$

is the *operator of differentiation by virtue of the Hamiltonian system* (1.1).

A function $F \in C^1(D)$ is called a *first integral on the domain* $D \subset \mathbb{R}^{2n+1}$ of the Hamiltonian system (1.1) if $\mathfrak{B}\, F(t,q,p) = 0$ or

$$\partial_t F(t,q,p) + \big[F(t,q,p), H(q,p)\big] = 0 \ \text{ for all } \ (t,q,p) \in D.$$

A function $F \in C^1(G)$ is an *autonomous first integral* of the Hamiltonian system (1.1) if the functions F and H are in involution, i.e., $[F(q,p), H(q,p)] = 0$ for all $(q,p) \in G \subset \mathbb{R}^{2n}$. Notice that the Hamiltonian H is an autonomous first integral of the Hamiltonian differential system (1.1).

A set of functionally independent on $D \subset \mathbb{R}^{2n+1}$ first integrals $F_l \in C^1(D)$, $l = 1, \dots, k$, of the Hamiltonian system (1.1) is called a *basis of first integrals* (or *integral basis*) on the domain D of system (1.1) if any first integral $F \in C^1(D)$ of system (1.1) can be represented on D in the form

$$F(t,q,p) = \Phi\big(F_1(t,q,p), \dots, F_k(t,q,p)\big) \ \text{ for all } \ (t,q,p) \in D,$$

where Φ is some continuously differentiable function. The number k is said to be the *dimension* of basis of first integrals on the domain D for the Hamiltonian differential system (1.1).

The Hamiltonian differential system (1.1) on an neighbourhood of any point from the domain D has a basis of first integrals of dimension $2n$ (see, for example, [4, p. 54]). Besides, the autonomous Hamiltonian differential system (1.1) on a domain G without equilibrium points has an autonomous integral basis of dimension $2n-1$ [1, pp. 167–169].

A polynomial w is a *partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[w(q,p), H(q,p)\big] = w(q,p)M(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \tag{1.2}$$

where the polynomial M (*cofactor* of the partial integral w) such that $\deg M \leqslant h-2$.

A partial integral w with cofactor M of the Hamiltonian system (1.1) is said to be *multiple* with multiplicity

$$\varkappa = 1 + \sum_{\xi=1}^{\varepsilon} r_\xi$$

if there exist natural numbers f_ξ and polynomials

$$Q_{f_\xi g_\xi}, \ \ g_\xi = 1, \dots, r_\xi, \ \ \xi = 1, \dots, \varepsilon,$$

such that on the domain $G \subset \{(q,p) : \ w(q,p) \neq 0\}$ the identities hold

$$\bigg[\frac{Q_{f_\xi g_\xi}(q,p)}{w^{f_\xi}(q,p)}\,, H(q,p)\bigg] = R_{f_\xi g_\xi}(q,p), \ \ g_\xi = 1, \dots, r_\xi, \ \ \xi = 1, \dots, \varepsilon, \tag{1.3}$$

where the polynomials $R_{f_\xi g_\xi}$ have degrees at most $h-2$. Note that a similar point of view on multiplicity of partial integrals was presented by J. Llibre and X. Zhang in [7].

An exponential function $\omega(q,p) = \exp v(q,p)$ for all $(q,p) \in \mathbb{R}^{2n}$ with some real polynomial v is called a *conditional partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[v(q,p), H(q,p)\big] = S(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \tag{1.4}$$

where the polynomial S (*cofactor* of the conditional partial integral ω) such that $\deg S \leqslant h-2$.

We stress that a conditional partial integral is a special case of exponential factor (or exponential partial integral) [2, 5, 6] for the polynomial Hamiltonian ordinary differential system (1.1).

2 Main results

The general results of this paper are formulated in Theorems 2.1–2.3.

Theorem 2.1. *If the Hamiltonian system* (1.1) *has the partial integral w with cofactor*

$$M(q,p) = \lambda \ \textit{for all} \ (q,p) \in \mathbb{R}^{2n}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \tag{2.1}$$

then an non-autonomous first integral of the autonomous Hamiltonian system (1.1) *is the function*

$$F(t,q,p) = \ w(q,p) \exp\bigl(-\lambda t\bigr) \ \textit{for all} \ (t,q,p) \in \mathbb{R}^{2n+1}.$$

Proof. Using the identity (1.2) under the condition (2.1), we have

$$\mathfrak{B} F(t,q,p) = \partial_t F(t,q,p) + \bigl[F(t,q,p), H(q,p)\bigr] = F(t,q,p)\, \partial_t\bigl(-\lambda t\bigr) \\ + \exp\bigl(-\lambda t\bigr)\bigl[w(q,p), H(q,p)\bigr] = 0 \ \text{ for all } \ (t,q,p) \in \mathbb{R}^{2n+1}.$$

Therefore the function F is a first integral of the autonomous Hamiltonian system (1.1). □

For example, the autonomous polynomial Hamiltonian differential system given by

$$H(q,p) = \frac{1}{2}\bigl(p_1^2 + p_2^2 - q_1^2 - q_2^2\bigr) \ \text{ for all } \ (q,p) \in \mathbb{R}^4 \tag{2.2}$$

has the polynomial partial integrals

$$w_1(q,p) = q_1 - p_1, \quad w_2(q,p) = q_2 - p_2, \quad w_3(q,p) = q_1 + p_1, \quad w_4(q,p) = q_2 + p_2$$

with cofactors

$$M_1(q,p) = M_2(q,p) = -1, \quad M_3(q,p) = M_4(q,p) = 1 \ \text{ for all } \ (q,p) \in \mathbb{R}^4.$$

By Theorem 2.1, we can build the non-autonomous first integrals of the Hamiltonian system (2.2)

$$F_1(t,q,p) = (q_1 - p_1)e^t, \quad F_2(t,q,p) = (q_2 - p_2)e^t, \\ F_3(t,q,p) = (q_1 + p_1)e^{-t}, \quad F_4(t,q,p) = (q_2 + p_2)e^{-t}.$$

The functionally independent non-autonomous first integrals $F_1, \ldots, F_4$ are an integral basis (non-autonomous) of the autonomous Hamiltonian system (2.2) on the space $\mathbb{R}^5$.

Theorem 2.2. *Suppose the polynomial Hamiltonian differential system* (1.1) *has the partial integral w with multiplicity*

$$\varkappa = 1 + \sum_{\xi=1}^{\varepsilon} r_\xi.$$

If the identity (1.3) *under some numbers $\xi \in \{1,\ldots,\varepsilon\}$ and $g_\xi \in \{1,\ldots,r_\xi\}$ such that the polynomial*

$$R_{f_\xi g_\xi}(q,p) = \lambda \ \textit{for all} \ (q,p) \in G \subset \mathbb{R}^{2n}, \quad \lambda \in \mathbb{R}, \tag{2.3}$$

then an non-autonomous first integral of the autonomous Hamiltonian system (1.1) *is the function*

$$F(t,q,p) = K_{f_\xi g_\xi}(q,p) \ - \ \lambda t \ \textit{for all} \ (t,q,p) \in \mathbb{R} \times G.$$

Proof. Taking into account the identity (1.3) under the condition (2.3), we obtain

$$\mathfrak{B}\,F(t,q,p)=\partial_t F(t,q,p)+\big[F(t,q,p),H(q,p)\big]=\,-\,\partial_t\big(\lambda t\big)+\big[K_{f_\xi g_\xi}(q,p),H(q,p)\big]=0.$$

For example, the autonomous polynomial Hamiltonian differential system given by [9]

$$H(q,p)=-\,q_1^2+6q_1q_2+(2p_1+p_2)q_1+2q_2p_2+3p_2^2\ \text{ for all }\ (q,p)\in\mathbb{R}^4 \tag{2.4}$$

has the multiple partial integral $w_1(q,p)=\,3q_1+2p_2$ for all $(q,p)\in\mathbb{R}^4$ with

$$M_1(q,p)=\,-\,2,\quad K_{1,11}(q,p)=\frac{17q_1+12q_2+8p_1}{32(3q_1+2p_2)}\,,\quad R_{1,11}(q,p)=1,$$

and the multiple partial integral $w_2(q,p)=q_1$ for all $(q,p)\in\mathbb{R}^4$ with

$$M_2(q,p)=2,\quad K_{2,11}(q,p)=\frac{2q_2+3p_2}{16q_1}\,,\quad R_{2,11}(q,p)=\,-\,1.$$

Using Theorems 2.1 and 2.2, we can construct the basis (non-autonomous) of first integrals on a domain $\mathbb{R}\times G,\ G\subset G_1\cap G_2$, for the autonomous polynomial Hamiltonian system (2.4)

$$F_1(t,q,p)=(3q_1+2p_2)\,e^{2t},\quad F_2(t,q,p)=\frac{17q_1+12q_2+8p_1}{16(3q_1+2p_2)}-t,\quad G_1\subset\big\{(q,p):\ 3q_1+2p_2\neq 0\big\},$$
$$F_3(t,q,p)=q_1\,e^{-2t},\quad F_4(t,q,p)=\frac{2q_2+3p_2}{8q_1}+t,\quad G_2\subset\big\{(q,p):\ q_1\neq 0\big\}.$$

Notice also that the functionally independent autonomous first integrals (see [9])

$$W_1(q,p)=(3q_1+2p_2)\exp\Big(\frac{17q_1+12q_2+8p_1}{16(3q_1+2p_2)}\Big)\ \text{ for all }\ (q,p)\in G_1,$$
$$W_2(q,p)=q_1\exp\Big(\frac{2q_2+3p_2}{8q_1}\Big)\ \text{ for all }\ (q,p)\in G_2,$$
$$W_3(q,p)=\frac{17q_1+12q_2+8p_1}{32(3q_1+2p_2)}+\frac{2q_2+3p_2}{16q_1}\ \text{ for all }\ (q,p)\in G$$

of system (2.4) are an autonomous integral basis of the Hamiltonian system (2.4) on any domain G. □

Theorem 2.3. *Suppose the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integral* ω. *If the identity* (1.4) *such that the polynomial*

$$S(q,p)=\lambda\ \textit{ for all }\ (q,p)\in\mathbb{R}^{2n},\ \ \lambda\in\mathbb{R}\setminus\{0\}, \tag{2.5}$$

then the Hamiltonian system (1.1) *has the non-autonomous first integral*

$$F(t,q,p)=v(q,p)-\lambda t\ \textit{ for all }\ (t,q,p)\in\mathbb{R}^{2n+1}.$$

Proof. Using the identity (1.4) under the condition (2.5), we get

$$\mathfrak{B}\,F(t,q,p)=\partial_t F(t,q,p)+\big[F(t,q,p),H(q,p)\big]=\partial_t\big(-\,\lambda t\big)+\big[v(q,p),H(q,p)\big]=0.$$ □

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