Optimal Conditions for the Solvability of the Cauchy Weighted Problem for Higher Order Singular in Time and Phase Variables Ordinary Differential Equations

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The theory of the Cauchy problem for ordinary differential equations and systems with non-integrable singularities in the time variable was constructed in the early 1970s (see, e.g., [1] and the references therein). However, the investigation of this problem for singular in phase variables differential equations was started later (see [2]). In [3], unimprovable in a certain sense conditions are established guaranteeing, respectively, the solvability, unique solvability and unsolvability of the Cauchy weighted problem for singular in time and phase variables ordinary delayed differential equations. The results below are refinements of the theorems proved in [3] on the solvability and unsolvability of the Cauchy weighted problem for differential equations without delay.

We use the following notation.

$\mu! = 1$ for $\mu \in]-1,0]$ and $\mu! = \prod\limits_{i=0}^{m}(i+\mu_0)$ for $\mu = m+\mu_0$, where $\mu_0 \in]0,1[$ and m is a nonnegative integer;

$\mathbb{R}_+ = [0,+\infty[\,$, $\mathbb{R}_{0+} =]0,+\infty[\,$;

If n is a natural number, $\alpha \in \mathbb{R}_{0+}$, $x \in \mathbb{R}_{0+}$, and $q :]a,b[\, \to \mathbb{R}_+$ is a continuous function, satisfying the condition

$$\int\limits_a^t q(s)\,ds < +\infty \ \text{ for } \ a < t < b,$$

then

$$\mathcal{D}_*^{n,\alpha}(\,]a,b[\,;x) = \Big\{(t,x_1,\dots,x_n) \in]a,b[\,\times\mathbb{R}_{0+}^n : \ \ x_i \geq \frac{\alpha!}{(n-i+\alpha)!}\,(t-a)^{n-i+\alpha}x \ \ (i=1,\dots,n)\Big\},$$

$$\mathcal{D}^{n,\alpha}(\,]a,b[\,;x;q) = \Big\{(t,x_1,\dots,x_n) \in]a,b[\,\times\mathbb{R}_{0+}^n : \ \ Q^{(i-1)}(t) \leq x_i \leq \frac{\alpha!}{(n-i+\alpha)!}\,(t-a)^{n-i+\alpha}x \ \ (i=1,\dots,n)\Big\},$$

where

$$Q(t) = \frac{1}{(n-i)!}\int\limits_a^t (t-s)^{n-1}q(s)\,ds.$$

Consider the differential equation

$$u^{(n)} = f(t,u,\dots,u^{(n-1)}) \tag{1}$$

with the weighted initial conditions

$$\limsup_{t\to a} \frac{u^{(i-1)}(t)}{(t-a)^{n-i+\alpha}} < +\infty \ \ (i=1,\dots,n), \tag{2}$$

where $f :]a,b[\, \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is a continuous function, and α is a positive constant.

We are interested in the case where the function f has singularities in both time and phase variables, i.e. the case, where

$$\int_a^t f(s,x_1,\dots,x_n)\,ds = +\infty \ \text{ for } \ a<t<b, \ \ (x_1,\dots,x_n)\in\mathbb{R}^n_{0+},$$

$$\lim_{x_1+\cdots+x_n\to 0} f(t,x_1,\dots,x_n) = +\infty \ \text{ for } \ a<t<b.$$

By a solution of Eq. (1) it is naturally understood an n-times continuously differentiable function $u :]a,b[\, \to \mathbb{R}$, satisfying this equation together with the inequalities

$$u^{(i-1)}(t) > 0 \ \ (i=1,\dots,n)$$

in the interval $]a,b[$.

Theorem 1. *Let the function f in the domain $]a,b[\, \times \mathbb{R}^n_{0+}$ admit the estimate*

$$f(t,x_1,\dots,x_n) \geq q(t),$$

where $q :]a,b[\, \to \mathbb{R}_+$ is a continuous function, satisfying the condition

$$x_0 = \limsup_{t\to a} \bigg((t-a)^{-\alpha} \int_a^t q(s)\,ds \bigg) < +\infty.$$

Let, moreover, there exist continuous functions p and $q_0 :]a,b[\, \to \mathbb{R}_+$ such that

$$\limsup_{t\to a} \bigg((t-a)^{-\alpha} \int_a^t p(s)\,ds \bigg) < 1, \quad \limsup_{t\to a} \bigg((t-a)^{-\alpha} \int_a^t q_0(s)\,ds \bigg) < +\infty,$$

and on the set $\mathcal{D}^{n,\alpha}(]a,b[\,;x;q)$ the inequality

$$f(t,x_1,\dots,x_n) \leq p(t)x + q_0(t)$$

holds for any $x > x_0$. Then problem (1), (2) has at least one solution.

The restrictions imposed on the function f in the above theorem are optimal in a certain sense. The following theorem is valid.

Theorem 2. *Let the function f in the domain $\mathcal{D}^{n,\alpha}_*(]a,b[\,;x)$ admit the estimate*

$$f(t,x_1,\dots,x_n) \geq p(t)x + q(t),$$

where p and $q :]a,b[\, \to \mathbb{R}_+$ are continuous functions, satisfying the conditions

$$\int_a^t p(s)\,ds < +\infty, \ \ \int_a^t q(s)\,ds < +\infty \ \text{ for } \ a<t<b, \quad \liminf_{t\to a} \bigg((t-a)^{-\alpha} \int_a^t q(s)\,ds \bigg) > 0.$$

Let, moreover, either

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t q(s)\,ds\bigg)=+\infty,$$

or there exist $b_0\in\,]a,b[$ *such that*

$$\int\limits_a^t p(s)\,ds\geq(t-a)^{\alpha}\ \ \textit{for}\ \ a\leq t\leq b_0.$$

Then problem (1), (2) *has no solution.*

The two corollaries below of Theorems 1 and 2 concern the case where the function f in the domain $]a,b[\,\times\mathbb{R}^n_{0+}$ admits one of the following two estimates

$$q(t)\leq f(t,x_1,\dots,x_n)\leq\sum_{i=1}^m\Big(p_i(t)\prod_{k=1}^n x_k^{\gamma_{ik}}+q_i(t)\prod_{k=1}^n x_k^{-\lambda_{ik}}\Big)+q_0(t),\tag{3}$$

$$f(t,x_1,\dots,x_n)\geq\sum_{i=1}^m\Big(p_i(t)\prod_{k=1}^n x_k^{\gamma_{ik}}+q_i(t)\prod_{k=1}^n x_k^{-\lambda_{ik}}\Big)+q(t),\tag{4}$$

or Eq. (1) has the form

$$u^{(n)}=\sum_{i=1}^m\Big(p_i(t)\prod_{k=1}^n (u^{(k-i)})^{\gamma_{ik}}+q_i(t)\prod_{k=1}^n (u^{(k-i)})^{-\lambda_{ik}}\Big)+q_0(t).\tag{5}$$

Here and in what follows we assume that m is an arbitrary natural number, γ_{ik}, λ_{ik} $(i=1,\dots,m;$ $k=1,\dots,n)$ are nonnegative constants, satisfying the conditions

$$\sum_{k=1}^n\gamma_{ik}=1,\quad \sum_{k=1}^n\lambda_{ik}>0\ \ (i=1,\dots,m),$$

and $p_i:\,]a,b[\,\to\mathbb{R}_+$ $(i=1,\dots,m)$, $q_j:\,]a,b[\,\to\mathbb{R}_+$ $(j=0,\dots,m)$ and $q:\,]a,b[\,\to\mathbb{R}_+$ are continuous functions.

Let

$$\ell_i=\prod_{k=1}^n\Big(\frac{\alpha!}{(n-k+\alpha)!}\Big)^{\gamma_{ik}},\quad \mu_i=\sum_{k=1}^n(n-k+\alpha)\gamma_{ik},\quad \nu_i=\sum_{k=1}^n(n-k+\alpha)\lambda_{ik}\ \ (i=1,\dots,m).$$

Corollary 1. *If along with estimate* (3) *the conditions*

$$\limsup_{t\to a}\bigg(\sum_{i=1}^m\ell_i(t-a)^{-\alpha}\int\limits_a^t(s-a)^{\mu_i}p_i(s)\,ds\bigg)<1,\tag{6}$$

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t\Big(q_0(s)+\sum_{i=1}^m(s-a)^{-\nu_i}q_i(s)\Big)\,ds\bigg)<+\infty,\tag{7}$$

$$\liminf_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t q(s)\,ds\bigg)>0,\quad \limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t q(s)\,ds\bigg)<+\infty$$

hold, then problem (1), (2) *has at least one solution.*

Corollary 2. *Let the function f admit estimate (4) and let, moreover, either the condition*

$$\limsup_{t\to a}\left((t-a)^{-\alpha}\int_a^t\Big(q_0(s)+\sum_{i=1}^m(s-a)^{-\nu_i}q_i(s)\Big)\,ds\right)=+\infty \tag{8}$$

hold or there exist numbers $b_0\in\,]a,b[\,$, $\delta>0$ such that in the interval $]a,b_0[$ the following inequalities are satisfied:

$$\sum_{i=1}^m\ell_i\int_a^t(s-a)^{\mu_i}p_i(s)\,ds\geq(t-a)^{\alpha},\qquad \int_a^t q_0(s)\,ds\geq\delta(t-a). \tag{9}$$

Then problem (1), (2) has no solution.

The above corollaries imply the following statements for problem (5), (2).

Corollary 3. *If along with inequalities (6), (7), the inequality*

$$\liminf_{t\to a}\left((t-a)^{-\alpha}\int_a^t q_0(s)\,ds\right)>0 \tag{10}$$

holds, then problem (5), (2) has at least one solution. If condition (8) is satisfied or for some $b_0\in\,]a,b[$ and $\delta>0$ inequalities (9) hold, then problem (5), (2) has no solution.

Corollary 4. *Let inequality (10) hold and let there exist numbers $b_0\in\,]a,b[$ and $\ell\geq0$ such that in the interval $]a,b[$ the following equality*

$$\sum_{i=1}^m\ell_i(t-a)^{\mu_i}p_i(t)=\ell(t-a)^{\alpha-1}$$

is satisfied. Then for the solvability of problem (5), (2) it is necessary and sufficient that, along with (7), the condition

$$\ell<\alpha$$

be satisfied.

Remark. Let

$$p_i(t)\equiv p_{i0}(t-a)^{-\sum\limits_{k=1}^n(n-k)\gamma_{ik}-1},\quad q_i(t)\equiv q_{i0}(t-a)^{\nu_{0i}}\ \ (i=1,\dots,m),$$
$$q(t)=q_{00}(t-a)^{\alpha-1},$$

where p_{i0}, q_{i0} $(i=1,\dots,m)$, q_{00} are positive constants and

$$\nu_{0i}>\nu-1\ \ (i=1,\dots,m).$$

Then, according to Corollary 4, for the solvability of problem (5), (2) it is necessary and sufficient that the inequality

$$\sum_{i=1}^m\ell_i p_{0i}<\alpha$$

be satisfied.

References

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