Nonlinear Autonomous Boundary-Value Problem for Differential Algebraic System

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We suppose that A and B are $(m\times n)$-measurable matrices and $Z(z,\varepsilon)$ is an n measurable vector function. We will call a weakly nonlinear autonomous periodic differential-algebraic boundary-value problem the problem of finding solutions [6]

$$z(t,\varepsilon):\ \ z(\,\cdot\,,\varepsilon)\in\mathbb{C}^1[a,b(\varepsilon)],\ \ z(t,\,\cdot\,)\in\mathbb{C}[0,\varepsilon_0],\ \ b(0):=b^*$$

of the differential-algebraic system

$$A\,z'=B\,z+\varepsilon Z(z,\varepsilon), \tag{1}$$

satisfying the boundary condition

$$\ell z(\,\cdot\,,\varepsilon)=\alpha. \tag{2}$$

Here, $\ell z(\,\cdot\,,\varepsilon)$ is a linear bounded vector functional

$$\ell z(\,\cdot\,,\varepsilon):\mathbb{C}[a,b(\varepsilon)]\to\mathbb{R}^q.$$

We seek solutions of problem $(1),(2)$ in a small neighborhood of the solution $z_0(t)\in\mathbb{C}^1[a,b^*]$ of the generating Noether $(q\neq n)$ differential-algebraic boundary-value problem

$$A\,z_0'=B\,z_0,\ \ \ell z_0(\,\cdot\,)=\alpha\in\mathbb{R}^q. \tag{3}$$

We assume that the vector function $Z(z,\varepsilon)$ is a continuously differentiable with respect to the unknown $z(t,\varepsilon)$ in a small neighborhood of the solution of the generating problem and continuously differentiable with respect to the small parameter ε in a small positive neighborhood of zero. The matrix A is generally assumed to be rectangular $m\neq n$, or square, but degenerate. Under the condition

$$P_{A^*}=0 \tag{4}$$

the generating system (3) is reduced to the traditional system of ordinary differential equations [2]

$$z_0'=A^+B\,z_0+P_{A_{\rho_0}}\nu_0(t). \tag{5}$$

Moreover, A^+ is a pseudoinverse (by Moore–Penrose) matrix, P_{A^*} is a matrix orthoprojector

$$P_{A^*}:\mathbb{R}^m\to\mathbb{N}(A^*),$$

$P_{A_{\rho_0}}$ is an $(n \times \rho_0)$ matrix formed by ρ_0 linearly independent columns of the $(n \times n)$ matrix orthoprojector

$$P_A : \mathbb{R}^n \to \mathbb{N}(A),$$

$\nu_0(t) \in \mathbb{R}^{\rho_0}$ is an arbitrary continuous vector function. Under the condition (4) system (1) will be called nondegenerate. Suppose that the boundary-value problem for system (3) corresponds to a critical case

$$P_{Q^*} \neq 0, \quad Q := \ell X_0(\,\cdot\,).$$

In the critical case for a fixed vector function $\nu_0(t) \in \mathbb{C}[a, b^*]$ under the condition

$$P_{Q_d^*}\Big\{\alpha - \ell K\big[P_{A_{\rho_0}}\nu_0(s)\big](\,\cdot\,)\Big\} = 0 \tag{6}$$

the generating problem (3) has an r parametric family of solutions [3]

$$z_0(t, c_r) = X_r(t)c_r + G\big[P_{A_{\rho_0}}\nu_0(s)\big](t), \quad c_r \in \mathbb{R}^r.$$

Here, $X_0(t)$ is the normal $(X_0(a) = I_n)$ fundamental matrix of the homogeneous part of the differential system (5). Moreover,

$$G\big[P_{A_{\rho_0}}\nu_0(s)\big](t) := X_0(t)Q^+\ell K\big[P_{A_{\rho_0}}\nu_0(s)\big](\,\cdot\,) + K\big[P_{A_{\rho_0}}\nu_0(s)\big](t)$$

is the generalized Green's operator of the generating periodic differential-algebraic boundary-value problem (3) and

$$K\big[P_{A_{\rho_0}}\nu_0(s)\big](t) := X_0(t)\int_a^t X_0^{-1}(s)\,P_{A_{\rho_0}}\nu_0(s)\,ds$$

is the generalized Green's operator of the Cauchy problem $z(a) = 0$ for the differential-algebraic system (3). The matrix $P_{Q_d^*}$ formed by d linearly independent rows of the matrix orthoprojector P_{Q^*}, and the matrix P_{Q_r} formed by r linearly independent columns of the matrix orthoprojector P_Q. Under condition (4) system (1) is reduced to the traditional system of the ordinary differential equations

$$z' = A^+B\,z + P_{A_{\rho_0}}\nu_0(t) + \varepsilon\,A^+Z(z, \varepsilon). \tag{7}$$

The boundary-value problem for the nondegenerate differential-algebraic system (6) differs significantly from similar nonautonomous boundary-value problems depending on an arbitrary vector function $\nu_0(t) \in \mathbb{C}[a, b^*]$. In exceptional cases, the autonomous boundary-value problem (1), (2) is solvable on a segment of fixed length.

As is known [7], an autonomous boundary-value problem for system (7) differs significantly from similar nonautonomous boundary-value problems. Unlike the latter, the right end $b(\varepsilon)$ of the interval $[a, b(\varepsilon)]$, on which we are finding solution of the nonlinear boundary-value problem for system (7), is unknown and must be defined in the process of constructing the solution itself. Let's use the technique [6, 7] which consists in defining the unknown function

$$b(\varepsilon) = b^* + \varepsilon(b^* - a)\beta(\varepsilon)$$

in terms of the new unknown

$$\beta(\varepsilon) \in \mathbb{C}[0, \varepsilon_0], \quad \beta(0) := \beta^*.$$

The function $\beta(\varepsilon)$ is to be determined in the process of finding a solution of the boundary-value problem for system (7). The essence of the reception is to replace the independent variable

$$t = a + (\tau - a)\big(1 + \varepsilon\beta(\varepsilon)\big)$$

and finding a solution for the nonlinear boundary-value problem (2), (7) and the function $\beta(\varepsilon)$ as a function of a small parameter. In the critical case, under the condition (6) for a fixed function $\nu_0(\tau)$ the condition of solving of the nonlinear boundary-value problem (2), (7) takes the form [6]

$$P_{Q_d^*}\Big\{(1+\varepsilon\beta(\varepsilon))\,\alpha-\ell K\Big[\beta(\varepsilon)\big(A^+B\,z(s,\varepsilon)+P_{A_{\rho_0}}\nu_0(s)\big)+(1+\varepsilon\beta(\varepsilon))\,A^+Z(z(s,\varepsilon),\varepsilon)\Big](\,\cdot\,)\Big\}=0. \quad (8)$$

Using the continuously of the nonlinear vector function $Z(z(t,\varepsilon),\varepsilon)$ on ε in a small positive neighborhood of zero, we pass to the boundary for $\varepsilon\to 0$ in equality (8) and obtain the necessary condition

$$F(\check{c}_0):=P_{Q_d^*}\Big\{\alpha-\ell K\Big[\beta^*\big(A^+B\,z_0(s,c_r^*)+P_{A_{\rho_0}}\nu_0(s)\big)+A^+Z(z_0(s,c_r^*),0)\Big](\,\cdot\,)\Big\}=0 \qquad (9)$$

for the existence of a solution of the boundary-value problem (1), (2) in a critical case. Here,

$$\check{c}_0:=\begin{pmatrix} c_r^* \\ \beta^* \end{pmatrix}\in\mathbb{R}^{r+1}.$$

Thus, the following lemma is proved.

Lemma. *Suppose that the autonomous differential-algebraic boundary-value problem* (1), (2) *for a fixed constant $\nu_0\in\mathbb{R}^{\rho_0}$ under conditions* (4) *and* (6) *corresponds to the critical case $P_{Q^*}\neq 0$ and has the solution $z(t,\varepsilon)$, that for $\varepsilon=0$ is transformed into generating $z(t,0)=z_0(t,c_r^*)$. Then the vector $\check{c}_0$ satisfies to equation* (9)*.*

The first r components $c_r^*\in\mathbb{R}^r$ of the root of equation (9) determine the amplitude of the generating solution $z_0(t,c_r^*)$ in a small neighborhood of which can exist the desired solution of the original problem (1), (2). In addition, from equation (9) can be found the value β^* which determines the first approximation to the unknown function

$$b_1(\varepsilon)=b^*+\varepsilon(b^*-a)\beta^*.$$

If equation (9) has no real roots, then the original differential-algebraic problem (1), (2) does not have the desired solutions. Equation (9) will be further called the equation for generating constants of the autonomous nonlinear differential-algebraic boundary-value problem (1), (2). The statement of the lemma generalizes the corresponding results of [1, 5] onto the case of the autonomous nonlinear differential-algebraic boundary-value problem (1), (2), namely, for the case of $A\neq I_n$. As is known [1, 5, 6], the nondegenerate differential-algebraic problem (1) (2) is solvable when the roots of the equation for generating constants (9) are simple. Proposed in the article scheme of study of the nonlinear autonomous boundary-value problem for a nondegenerate system of differential-algebraic equations can be transferred, analogously to [3], onto degenerate systems of differential-algebraic equations. The above-proposed scheme of study of the nonlinear autonomous boundary value problem for a nondegenerate system of differential-algebraic equations can be transferred, analogously to [4], onto systems of differential-algebraic equations with a matrix of variable rank at the derivative, and analogously to [8], onto nonlinear boundary-value problems not solved with respect to the derivative.

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