Two Point Boundary Value Problems for the Fourth Order Ordinary Differential Equations

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We study on the interval $I := [a, b]$ the fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t), \tag{0.1}$$

and

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)) + h(t), \tag{0.2}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \ \ (j = 0, 1), \tag{0.3_1}$$

$$u^{(j)}(a) = 0 \ \ (j = 0, 1, 2), \quad u(b) = 0, \tag{0.3_2}$$

where $p, h \in L(I; \mathbb{R})$, $f \in K(I \times \mathbb{R}; \mathbb{R})$.

By a solution of problem $(0.2), (1.3_i)$ $(i \in \{1, 2\})$ we understand a function $u \in \widetilde{C}^3(I; \mathbb{R})$, which satisfies equation (0.2) a.e. on I, and conditions (1.3_i).

Throughout the paper we use the following notations.

$C(I; \mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm

$$\|u\|_C = \max\big\{|u(t)| : \ t \in I\big\}.$$

$\widetilde{C}^{(3)}(I; \mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm

$$\|p\|_L = \int\limits_a^b |p(s)|\, ds.$$

$K(I \times \mathbb{R}; \mathbb{R})$ is the set of functions $f : I \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions, i.e., $f(\,\cdot\,, x) : I \to \mathbb{R}$ is a measurable function for all $x \in \mathbb{R}$, $f(t, \cdot\,) : \mathbb{R} \to \mathbb{R}$ is a continuous function for almost all $t \in I$, and for arbitrary $r > 0$ the inclusion

$$f^*_r(t) := \sup\big\{|f(t, x)| : \ \ |x| \le r\big\} \in L(I; \mathbb{R}^+_0)$$

holds.

For arbitrary $x, y \in L(I;\mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \ (x(t) \succcurlyeq y(t)) \ \text{ for } \ t \in I,$$

means that $x \le y$ $(x \ge y)$ and $x \ne y$.

We also use the notation $[x]_\pm = (|x| \pm x)/2$.

The aim of our work is to study the solvability of the above mentioned problems. We have proved the unimprovable sufficient conditions of the unique solvability for the linear problem, which show that the solvability of problem $(0.1), (0.3_1)$ $((0.1), (0.3_2))$ depends only on the nonnegative (non positive) part of the coefficient p if this nonnegative (non positive) part is small enough. On the basis of these results for the nonlinear problems, sufficient conditions of solvability have been proved in non resonance and resonance cases in which nonlinearities can have the linear growth.

Below we present some definitions from the work [2] which we need for the formulation of our results.

Definition 0.1. Equation

$$u^{(4)}(t) = p(t)u(t) \ \text{ for } \ t \in I \tag{0.4}$$

is said to be disconjugate (non-oscillatory) on I if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 0.2. We will say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and problem $(0.4), (0.3_1)$ has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a,b[\,. \tag{0.5}$$

Definition 0.3. We will say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and problem $(0.4), (0.3_2)$ has a solution u such that inequality (0.5) holds.

1 Linear problems

The proofs of the following results of the unique solvability of problems $(0.1), (0.3_1)$ and $(0.1), (0.3_2)$ are based on the results from the papers [1] and [2].

Theorem 1.1. *Let $i \in \{1,2\}$ and the function $p_0 \in L(I;\mathbb{R})$ be such that the equation*

$$u^{(4)}(t) = [p_0(t)]_+ u(t) \ \textit{ if } \ i = 1,$$
$$u^{(4)}(t) = -[p_0(t)]_- u(t) \ \textit{ if } \ i = 2,$$

is diconjugate on I. Then if the inequality

$$(-1)^{i-1}[p(t) - p_0(t)] \le 0 \ \textit{ for } \ t \in I$$

holds, problem $(0.1), (0.3_i)$ is uniquely solvable.

From the last theorem with $p_0 = [p]_+$ or $p_0 = -[p]_-$ it immediately follows:

Corollary 1.1. *Let there exist $p^* \in D_+(I)$ $(p_* \in D_-(I))$ such that the inequality*

$$[p(t)]_+ \preccurlyeq p^*(t) \quad (\,-[p(t)]_- \succcurlyeq p_*(t)\,) \ \textit{ for } \ t \in I \tag{1.1}$$

holds. Then problem $(0.1), (0.3_1)$ $((0.1), (0.3_2))$ is uniquely solvable.

Remark. Condition (1.1) in Corollary 1.1 is optimal in the sense that the inequality $\preccurlyeq$ $(\succcurlyeq)$ can not be replaced by the inequality $\le$ $(\ge)$.

2 Nonlinear Problem at the non resonance case

On the basis of our results for the linear problems for the nonlinear problems in non resonance case, i.e. when problem $(0.4),(0.3_i)$ has only the trivial solution, in [3] we have proved the following solvability theorem:

Theorem 2.1. *Let $i \in \{1,2\}$ and there exist $r \in \mathbb{R}^+$ and $g \in L(I;\mathbb{R}_0^+)$ such that a.e. on I the inequality*

$$-g(t)|x| \le (-1)^{i-1} f(t,x) \operatorname{sgn} x \le \delta(t,|x|) \ \text{ for } \ |x| > r$$

holds, where the function $\delta \in K(I \times \mathbb{R}_0^+;\mathbb{R}_0^+)$ is nondecreasing in the second argument and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \delta(s,\rho)\,ds = 0.$$

Then if the equation

$$u^{(4)}(t) = [p(t)]_+ u(t) \ \text{ if } \ i = 1,$$
$$u^{(4)}(t) = -[p(t)]_- u(t) \ \text{ if } \ i = 2$$

is disconjugate, problem $(0.2),(0.3_i)$ has at least one solution.

3 Nonlinear Problem at the resonance

On the basis of Corollary 1.1 and Theorem 2.1 we proved the following Landesman–Laser type sufficient conditions of solvability of problem $(0.4),(0.3_i)$ at the resonance case. It is well known that problem $(0.4),(0.3_i)$ is unique solvable if $(-1)^{i+1}p(t) \le 0$. Therefor when we speak about problem (0.2), (0.3_i) at the resonance case we must assume that the condition

$$(-1)^{i+1} p(t) > 0 \ \text{ for } \ t \in I \tag{3.1}$$

holds.

Theorem 3.1. *Let $i \in \{1,2\}$ the constant $r > 0$ and the functions $f^-, f^+, g \in L(I;\mathbb{R}_0^+)$, $p \in L(I;\mathbb{R})$, be such that the conditions* (3.1),

$$p \in D_+(I) \ \text{ if } \ i = 1, \quad p \in D_-(I) \ \text{ if } \ i = 2, \tag{3.2}$$

and

$$f^-(t) \le (-1)^{i-1} f(t,x) \le g(t)|x| \ \text{ for } \ x < -r, \ \ t \in I,$$
$$-g(t)|x| \le (-1)^{i-1} f(t,x) \le -f^+(t) \ \text{ for } \ x > r, \ \ t \in I$$

hold. Moreover, let w be a nontrivial solution of homogeneous problem $(0.4),(0.3_i)$ and there exists $\varepsilon > 0$ such that the condition

$$-\int_a^b f^-(s)|w(s)|\,ds + \varepsilon\gamma_r \|w\|_C \le (-1)^{i-1} \int_a^b h(s)|w(s)|\,ds \le \int_a^b f^+(s)|w(s)|\,ds - \varepsilon\gamma_r\|w\|_C,$$

holds, where

$$\gamma_r = \int_a^b f_r^*(s)\,ds.$$

Then for an arbitrary function $h \in L(I;\mathbb{R})$ problem $(0.2),(0.3_i)$ is solvable.

Also is true the following existence and uniqueness theorem.

Theorem 3.2. *Let* $i \in \{1, 2\}$*, condition* (3.1)*,* (3.2) *holds and* $f(t, 0) \equiv 0$*. Moreover, let there exist functions* $\eta : \mathbb{R}^2 \to]0, +\infty[$ *, and* $g, \ell \in L(I; \mathbb{R}_0^+)$ *such that* $\ell \not\equiv 0$ *and the condition*

$$-g(t)|x_1 - x_2| \leq (-1)^{i-1}(f(t, x_1) - f(t, x_2)) \operatorname{sgn}(x_1 - x_2) \leq -\ell(t)\eta(x_1, x_2)|x_1 - x_2|,$$

for $t \in I$*,* $x_1, x_2 \in \mathbb{R}$ *holds, where*

$$\lim_{|\rho| \to +\infty} |\rho|\eta(\rho, 0) = +\infty.$$

Then for an arbitrary function $h \in L(I; \mathbb{R})$ *problem* $(0.2), (0.3_i)$ *is uniquely solvable.*

References

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[2] M. Manjikashvili and S. Mukhigulashvili, Necessary and sufficient conditions of disconjugacy for the fourth order linear ordinary differential equations. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **64(112)** (2021), no. 4, 341–353.

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