Robust Stability for the Attractors of Nonlinear Wave Equations

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1 Introduction

In the paper, we consider the qualitative behavior of a nonlinear wave equation with a non-smooth interaction function that recognizes external bounded disturbances. It is proved that the global attractor of the multivalued semiflow generated by the solutions of the undisturbed problem is stable in the sense of ISS with respect to the disturbances.

The qualitative behavior of infinite-dimensional evolutionary systems without uniqueness, i.e., when, along with global solvability, non-unity of the solution of the initial boundary value problem is also possible, began to be actively studied within the framework of the theory of attractors from the end of the 90s of the last century [9, 14, 15, 17, 21]. It turned out that for broad classes of evolutionary objects, under fairly general conditions for the parameters, it is possible to establish the existence in the phase space of a compact uniformly attracting set (be) the global attractor. Its stability in relation to disturbances has been studied in works [1–4, 7, 8, 10, 12]. The theory of input to state stability (ISS), which characterizes the deviation of solutions of a perturbed problem from an asymptotically stable equilibrium position [6, 16, 19, 20], was applied to infinite-dimensional dissipative systems with a nontrivial attractor in works [5, 11, 18]. In particular, the property of local input to state stability (local ISS) and the property of asymptotic gain (AG) for semi-linear parabolic and wave equations, provided that the Cauchy problem is correct, have been established.

In this paper, for the first time, the AG property was obtained for the global attractor of a dynamic system without uniqueness (m-semiflow), generated by the solutions of a nonlinear wave equation with a non-smooth interaction function.

2 Setting of the problem and the main results

In a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the problem

$$\begin{cases} y_{tt} + \alpha y_t - \triangle y + f(y) = 0, \quad t > 0, \\ y\big|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\alpha > 0$, $f \in \mathbb{C}(\mathbb{R})$,

$$\exists\, c > 0 \ \ \forall\, s \in \mathbb{R} \quad |f(s)| \le c\big(1 + |s|^{\frac{n}{n-2}}\big), \tag{2.2}$$

$$\varliminf_{s\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{2.3}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\triangle$ in $H_0^1(\Omega)$. Then it is known [1] that in the phase space

$$X = H_0^1(\Omega) \times L^2(\Omega)$$

problem (2.1) for every $z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X$ has a (perhaps non-unique) solution $z(\,\cdot\,) = \begin{pmatrix} y(\,\cdot\,) \\ y_t(\,\cdot\,) \end{pmatrix} \in \mathbb{C}([0,+\infty); X)$, $z(0) = z_0$, and all solutions (2.1) generate a multivalued semiflow (m-semiflow) $G : \mathbb{R}_+ \times X \mapsto 2^X$,

$$G(t, z_0) = \Big\{ z(t) : \ z(\,\cdot\,) \text{ is the solution of (2.1)}, \ z(0) = z_0 \Big\},$$

for which there is a global attractor in X.

Definition 2.1 ([14])**.** Let G be a m-semiflow, i.e.,

$$\forall\, x \in X, \ \ \forall\, t, s \geq 0 \quad G(0, x) = x, \ \ G(t+s, x) \subset G(t, G(s, t)).$$

A compact set $\Theta \subset X$ is called a global attractor G, if:

(1) $\Theta \subset G(t, \Theta)$ $\forall\, t \geq 0$;

(2) for any bounded set $B \subset X$,

$$\operatorname{dist}(G(t, B), \Theta) \to 0, \ \ t \to \infty,$$

where here and in the future

$$G(t, B) = \bigcup_{z \in B} G(t, z),$$
$$\operatorname{dist}(A, B) = \sup_{z_1 \in A} \inf_{z_2 \in B} \|z_1 - z_2\|_X.$$

Now consider the disturbed problem

$$\begin{cases} y_{tt} + \alpha y_t - \triangle y + f(y) = h(x) \cdot u(t), \quad t > 0, \\ y\big|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

where $h \in L^2(\Omega)$, $u \in L^\infty(0, +\infty)$ is the input (disturbing) signal.

Let's mark

$$S_u(t, 0, z_0) = \Big\{ z(t) : \ z(\,\cdot\,) \text{ is the solution of (2.4)}, \ z(0) = z_0 \Big\}.$$

The main result of the work is the establishment of the asymptotic gain (AG) property in relation to the attractor Θ of the unperturbed ($u \equiv 0$) system [18]:

$$\exists\, \gamma \in \mathcal{K} \ \ \forall\, z_0 \in X, \ \ \forall\, u \in U \subseteq L^\infty(0, +\infty) : \quad \overline{\lim_{t\to\infty}} \operatorname{dist}(S_u(t, 0, z_0), \Theta) \leq \gamma(\|u\|_\infty),$$

where U is some translationally invariant set of input signals, $\mathcal{K}$ is the class of continuous, monotonically increasing functions with $\gamma(0) = 0$ [13],

$$\|u\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} |u(t)|.$$

3 Robust stability and attractors of multivalued semiflows

Let $(X, \|\cdot\|_X)$ be the Banach space, $\mathbb{R}_{\geq} = \{(t,\tau): \ t \geq \tau \geq 0\}$, Σ be the arbitrary translation-invariant set, i.e.,

$$\forall\, \sigma \in \Sigma, \ \ \forall\, h \geq 0: \quad \sigma(\,\cdot\, + h) \in \Sigma.$$

Definition 3.1 ([2])**.** A family of multivalued mappings $\{S_\sigma: \ \mathbb{R}_{\geq} \times X \mapsto 2^X\}_{\sigma \in \Sigma}$ is called a family of m-semi-processes if $\forall\, \sigma \in \Sigma$, $\forall\, x \in X$, $\forall\, t \geq s \geq \tau \geq 0$, $\forall\, h \geq 0$:

$$\begin{gathered} S_\sigma(\tau,\tau,x) = x, \\ S_\sigma(t,\tau,x) \subset S_\sigma(t,s,S_\sigma(s,\tau,x)), \\ S_\sigma(t+h,\tau+h,x) \subset S_{\sigma(\,\cdot\,+h)}(t,\tau,x). \end{gathered}$$

Let's mark

$$S_\Sigma = \bigcup_{\sigma \in \Sigma} S_\sigma.$$

Definition 3.2 ([2])**.** A compact set $\Theta_\Sigma \subset X$ is called a uniform attractor $\{S_\sigma\}_{\sigma\in\Sigma}$ if for any bounded set $B \subset X$,

$$\operatorname{dist}\big(S_\Sigma(t,0,B), \Theta_\Sigma\big) \to 0, \ \ t \to \infty$$

and Θ_Σ is the minimal set in the class of such sets.

Remark 3.1. If $\Sigma = \{0\}$, then for $G(t,x) := S_0(t,0,x)$ we have the properties:

$$\begin{gathered} G(0,x) = S_0(0,0,x) = x, \\ G(t+s,x) = S_0(t+s,0,x) \subset S_0(t+s,s,S_0(s,0,x)) \subset S_0(t,0,S_0(s,0,x)) = G(t,G(s,x)), \end{gathered}$$

so G is the m-semiflow.

The following lemma guarantees the existence of a uniform attractor in $\{S_\sigma\}_{\sigma\in\Sigma}$.

Lemma 3.1 ([2])**.** *Let $\{S_\sigma\}_{\sigma\in\Sigma}$ be the family of m-semi-processes, Σ be the translation-invariant subset of some metric space and the next conditions be fulfilled:*

(1) *there exists a bounded set $B_0 \subset X$ such that for any bounded set $B \subset X$ exists $T = T(B)$ such that*

$$\forall\, t \geq T \quad S_\Sigma(t,0,B) \subset B_0;$$

(2) $\forall\,\{\sigma_n\} \subset \Sigma$, $\forall\, t_n \nearrow \infty$, $\forall$ *limited sequence $\{x_n\} \subset X$ sequence $\{\xi_n \in S_{\sigma_n}(t_n,0,x_n)\}_{n\geq 1}$ is precompact.*

Then $\{S_\sigma\}_{\sigma\in\Sigma}$ has a uniform attractor Θ_Σ.

If, in addition, the next condition is satisfied:

(3) *the mapping $\Sigma \times X \ni (\sigma, x) \mapsto S_\sigma(t,0,x) \subset X$ has a closed graph, then*

$$\Theta_\Sigma \subset S_\Sigma(t,0,\Theta_\Sigma).$$

Remark 3.2. In condition 1) it can be assumed that $B_0 = \{x \in X | \ \|x\|_X \leq R_0\}$.

Remark 3.3. For the $\Sigma = \{0\}$ conditions 1)-3) have the form:

$$\forall\, t \geq T \quad G(t,B) \subset B_0,$$

every sequence $\xi_n \in G(t_n, B)$ is precompact, the mapping $x \mapsto G(t,x)$ has a closed graph; and guarantee [14] that $\Theta := \Theta_{\{0\}}$ is a global attractor m-semiflow G.

Theorem 3.1. *Let for each $u \in U \subset L^\infty(\mathbb{R}_+)$ there exist a translation-invariant set $\Sigma(u)$ such that the family m-semi-processes $\{S_\sigma\}_{\sigma\in\Sigma(u)}$ satisfies conditions (1)–(3) of Lemma 3.1,*

$$\Sigma(0)=\{0\}, \ \ \forall\, u\in U \ \ u\in\Sigma(u),$$

$\forall\, r_0>0$ there exists the set B_0 such that condition (1) of Lemma 3.1 is fulfilled $\forall\,\|u\|_\infty\le r_0$, i.e.,

$$\exists\, T=T(r_0,B) \ \ \forall\, t\ge T \qquad \bigcup_{\|u\|_\infty\le r_0} S_{\Sigma(u)}(t,0,B)\subset B_0, \tag{3.1}$$

and in addition, the next conditions are met

(1)
$$\|u_k\|_\infty\to 0, \ \ t_k\to\infty \implies \xi_k\in S_{\Sigma(u_k)}(t_k,0,B_0)$$

is precompact,

(2)
$$\|u_k\|_\infty\to 0, \ \ x_k\to x, \ \ \xi_k\in S_{\Sigma(u_k)}(t,0,x_k), \ \ \xi_k\to\xi \implies \xi\in S_0(t,0,x).$$

Then
$$\exists\,\gamma\in\mathcal{K} \ \ \forall\, x\in X, \ \ \forall\, u\in U \qquad \varlimsup_{t\to\infty}\operatorname{dist}(S_u(t,0,x),\Theta)\le\gamma(\|u\|_\infty).$$

4 Application for the disturbed wave equation

We consider a perturbed problem (2.4). Let's strengthen condition (2.3) to the following:

$\exists\, c_1,c_2,c_3>0$ such that for $F(s):=\int\limits_0^s f(p)\,dp$ for all $s\in\mathbb{R}$ next inequalities are fulfilled

$$F(s)\ge -ms^2-c_1, \quad f(s)\cdot s-c_2F(s)+ms^2\ge c_3, \tag{4.1}$$

where $m\in(0,\lambda_1)$ is small enough.

Under conditions (2.2), (4.1) it is known [2] that $\forall\,\tau\ge 0$, $\forall\, z_\tau\in X$, $\forall\, u\in L^2_{loc}(\mathbb{R}_+)$ problem (2.4) has at least one solution $z\in\mathbb{C}([\tau,+\infty);X):\ z(\tau)=z_\tau$. Moreover, the family of mappings $\{S_u:\ \mathbb{R}_\ge\times X\mapsto 2^X\}$ such that

$$S_u(t,\tau,z_\tau)=\Big\{z(t):\ z(\,\cdot\,) \text{ is the solution of (2.4) and } z(\tau)=z_\tau\Big\} \tag{4.2}$$

generates a family of m-semiprocesses for any translation-invariant $U\subset L^2_{loc}(\mathbb{R}_+)$. In addition, for every solution (2.4) $z=\begin{pmatrix} y\\ y_t\end{pmatrix}$ the next evaluation is fair:

$$\|y_t(t)\|^2_{L^2}+\|y(t)\|^2_{H^1_0}\le c_4\bigg(\Big(\|y_t(\tau)\|^2_{L^2}+\|y(\tau)\|^{\frac{2n-2}{n-2}}_{H^1_0}\Big)\cdot e^{-\delta(t-\tau)}+1+\int\limits_\tau^t |u(p)|^2e^{-\delta(t-p)}\,dp\bigg) \quad \forall\, t\ge\tau\ge 0,$$

where $c_4>0$, $\delta>0$ do not depend on z.

In particular, if $\sup\limits_{t\geq 0}\int\limits_t^{t+1}|u(p)|^2\,dp<\infty$, then $\forall\, t\geq\tau\geq 0$,

$$\|z(t)\|_X^2\leq c_5\left(\|z(\tau)\|_X^{\frac{2n-2}{n-2}}\cdot e^{-\delta(t-\tau)}+1+\sup_{t\geq 0}\int\limits_t^{t+1}|u(p)|^2\,dp\right). \tag{4.3}$$

As U, we choose all functions from $L^\infty(\mathbb{R}_+)$ for which

$$\sup_{t\geq 0}\int\limits_t^{t+1}|u(s+l)-u(s)|^2\,ds\leq\varkappa(|l|), \tag{4.4}$$

where $\varkappa$ may depend on u and $\varkappa(p)\to 0$, $p\to 0+$.

It is known [2] that $\forall\, u\in U$ the set

$$\Sigma(u):=cl_{L^2_{loc}}\{u(\,\cdot\,+h)|,\ h\geq 0\}$$

is translation invariant and compact in $L^2_{loc}(\mathbb{R}_+)$, $u\in\Sigma(u)$, $\Sigma(0)=\{0\}$ and, in addition,

$$\sup_{t\geq 0}\int\limits_t^{t+1}|v(s)|^2\,ds\leq\sup_{t\geq 0}\int\limits_t^{t+1}|u(s)|^2\,ds\leq\|u\|_\infty^2\ \ \forall\, v\in\Sigma(u). \tag{4.5}$$

If condition (4.4) is fulfilled, the family of m-semi-processes $\{S_v\}_{v\in\Sigma(u)}$, defined in (4.2), satisfies conditions (1)–(3) of Lemma 3.1, and therefore has a uniform attractor $\Theta_{\Sigma(u)}$. At the same time, due to (4.3) and (4.5), condition (3.1) is fulfilled.

Theorem 4.1. *Let the parameters of the disturbed problem* (2.4) *satisfy conditions* (2.2)*,* (4.1)*, and* (4.4)*. Then*

$$\exists\,\gamma\in\mathcal{K}\ \ \forall\, z_0\in X,\ \ \forall\, u\in U\qquad \varlimsup_{t\to\infty}\operatorname{dist}(S_u(t,0,z_0),\Theta)\leq\gamma(\|u\|_\infty),$$

where Θ *is the global attractor of the undisturbed problem* (2.1)*.*

Acknowledgements

This work is supported by the DAAD, Bilateral Exchange of Academics, 2022.

References

[1] J. M. Ball, Global attractors for damped semilinear wave equations. Partial differential equations and applications. *Discrete Contin. Dyn. Syst.* **10** (2004), no. 1-2, 31–52.

[2] V. V. Chepyzhov and M. I. Vishik, *Attractors for equations of Mathematical Physics*. American Mathematical Society Colloquium Publications, 49. American Mathematical Society, Providence, RI, 2002.

[3] S. Dashkovskiy, P. Feketa, O. Kapustyan and I. Romaniuk, Invariance and stability of global attractors for multi-valued impulsive dynamical systems. *J. Math. Anal. Appl.* **458** (2018), no. 1, 193–218.

[4] S. Dashkovskiy, O. Kapustyan and I. Romaniuk, Global attractors of impulsive parabolic inclusions. *Discrete Contin. Dyn. Syst. Ser. B* **22** (2017), no. 5, 1875–1886.

[5] S. Dashkovskiy, O. Kapustyan and J. Schmid, A local input-to-state stability result w.r.t. attractors of nonlinear reaction-diffusion equations. *Math. Control Signals Systems* **32** (2020), no. 3, 309–326.

[6] S. Dashkovskiy and A. Mironchenko, Input-to-state stability of infinite-dimensional control systems. *Math. Control Signals Systems* **25** (2013), no. 1, 1–35.

[7] N. V. Gorban, A. V. Kapustyan, E. A. Kapustyan and O. V. Khomenko, Strong global attractor for the three-dimensional Navier–Stokes system of equations in unbounded domain of channel type. *J. Aütom. Inf. Sci.* **47** (2015), no. 11, 48–59.

[8] O. V. Kapustyan, O. A. Kapustyan and A. V. Sukretna, Approximate stabilization for a nonlinear parabolic boundary-value problem. *Ukrainian Math. J.* **63** (2011), no. 5, 759–767.

[9] A. V. Kapustyan and V. S. Mel'nik, On the global attractors of multivalued semidynamical systems and their approximations. (Russian) *Dokl. Akad. Nauk* **366** (1999), no. 4, 445–448.

[10] O. V. Kapustyan and M. O. Perestyuk, Global attractors of impulsive infinite-dimensional systems. (Ukrainian) *Ukraïn. Mat. Zh.* **68** (2016), no. 4, 517–528; translation in *Ukrainian Math. J.* **68** (2016), no. 4, 583–597.

[11] O. V. Kapustyan and T. V. Yusypiv, Stability to disturbances for the attractor of the dissipative PDE ODE-type system. (Ukrainian) Nelīnīĭnī Koliv. **24** (2021), no. 3, 336–341.

[12] P. O. Kasyanov, V. S. Mel'nik and S. Toscano, Solutions of Cauchy and periodic problems for evolution inclusions with multi-valued w_{λ_0}-pseudomonotone maps. *J. Differential Equations* **249** (2010), no. 6, 1258–1287.

[13] H. K. Khalil, *Nonlinear Systems.* Third edition. Prentice Hall, New Jersey, 2002.

[14] V. S. Mel'nik, Multivalued dynamics of nonlinear infinite-dimensional systems. (Russian) Preprint, 94-17. *Natsional'naya Akademiya Nauk Ukrainy, Institut Kibernetiki im. V. M. Glushkova, Kiev*, 1994, 41 pp.

[15] V. S. Melnik and J. Valero, On attractors of multivalued semi-flows and differential inclusions. *Set-Valued Anal.* **6** (1998), no. 1, 83–111.

[16] A. Mironchenko, Local input-to-state stability: characterizations and counterexamples. *Systems Control Lett.* **87** (2016), 23–28.

[17] J. C. Robinson, *Infinite-Dimensional Dynamical Systems. An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001.

[18] J. Schmid, O. Kapustyan and S. Dashkovskiy, Asymptotic gain results for attractors of semilinear systems. *Math. Control Relat. Fields* **12** (2022), no. 3, 763–788.

[19] E. D. Sontag, Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control* **34** (1989), no. 4, 435–443.

[20] E. D. Sontag and Y. Wang, On characterizations of the input-to-state stability property. *Systems Control Lett.* **24** (1995), no. 5, 351–359.

[21] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Second edition. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997.