Global Stability of Nonlinear Delay Itô Equations and N. V. Azbelev's W-Method

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The classical stability analysis based on Lyapunov functions is the main tool in the theory of ordinary differential equations. However, applications of this method to functional differential equations often encounters serious difficulties. A successful alternative, known as the "N. V. Azbelev W-method", is based on searching auxiliary equations instead of Lyapunov functionals. The W-method is also efficient in studying various classes of stochastic delay differential equations.

However, application of the W-method to nonlinear functional equations remains less efficient, even if N .V. Azbelev and P. M. Simonov formulated some general results for nonlinear deterministic functional differential equations in their monograph [2].

In this work we study global Lyapunov stability of solutions of systems of nonlinear differential Itô equations with delays. We describe a nonlinear modification of the W-method based on the theory of inverse-positive matrices and provide sufficient conditions for the moment stability of solutions in terms of the coefficients for rather general classes of Itô equations.

Let $\mathcal{T}=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space.

We study the moment exponential stability of solutions to the following system of nonlinear Itô differential equations with delay:

$$\begin{aligned} dx(t)=&-\sum_{j=1}^{N}A^j(t)x(h_j(t))\,dt+F\big(t,x(h_1^0(t)),\dots,x(h_{m_0}^0(t))\big)\,dt\\ &+\sum_{i=1}^{m}G^i\big(t,x(h_1^i(t),\dots,x(h_{m_i}^i(t)))\big)\,d\mathcal{B}_i(t)\ \ (t\geq 0) \end{aligned}\tag{0.1}$$

with respect to the initial data

$$x(t)=\varphi(t)\ \ (t<0),\tag{0.1a}$$
$$x(0)=b,\tag{0.1b}$$

where $x=(x_1,\dots,x_n)^T$ is an unknown n-dimensional random process on the interval $(-\infty,\infty)$ called a solution to problem $(0.1),(0.1a),(0.1b)$.

We assume that problem $(0.1),(0.1a),(0.1b)$ satisfies the following

Conditions 1:

- $A^j = (a^j_{sl})^n_{s,l=1}$ are $n \times n$-matrices, whose entries are progressively measurable (with respect to the stochastic basis $\mathcal{T}$), scalar stochastic processes, the trajectories of which are almost surely (a.s.) locally integrable for all $j = 1, \dots, N$.

- $F(\,\cdot\,, u) = (F_1(\,\cdot\,, u_1, \dots, u_{m_0}), \dots, F_n(\,\cdot\,, u_1, \dots, u_{m_0}))^T$ are progressively measurable, n-dimensional stochastic processes on the interval $[0, \infty)$ with a.s. locally integrable trajectories for all $u \in R^{m_0}$, and $F(t, \,\cdot\,)$ are $P \times \mu$-almost everywhere continuous functions on R^{m_0}, satisfying the condition $F(\,\cdot\,, 0) = 0$.

- For all $i = 1, \dots, m$ the functions $G^i(\,\cdot\,, u) = (G^i_1(\,\cdot\,, u_1, \dots, u_{m_i}), \dots, G^i_n(\,\cdot\,, u_1, \dots, u_{m_i}))^T$ ($u \in R^{m_i}$) are progressively measurable, n-dimensional stochastic processes on the interval $[0, \infty)$ with a.s. locally square integrable trajectories, and $G^i(t, \,\cdot\,)$ are $P \times \mu$-almost everywhere continuous functions on R^{m_i}, satisfying the condition $G^i(\,\cdot\,, 0) = 0$.

- h_j, $j = 1, \dots, N$, h^i_j, $i = 0, \dots, m$, $j = 1, \dots, m_i$ are Borel measurable functions on $[0, \infty)$ such that $h_j(t) \le t$, $j = 1, \dots, N$, $h^i_j(t) \le t$, $i = 0, \dots, m$, $j = 1, \dots, m_i$ $(t \ge 0)$ μ-almost everywhere.

- φ is an $\mathcal{F}_0$-measurable n-dimensional stochastic process on the interval$(-\infty, 0)$.

- b is an $\mathcal{F}_0$-measurable n-dimensional random variable.

- For any initial conditions $(0.1a)$ and $(0.1b)$, which satisfy the above requirements, there exists a unique strong global solution $x(t, b, \varphi)$ to problem $(0.1), (0.1b)$, i.e., a solution defined on the initial stochastic basis and on the whole interval $(-\infty, \infty)$.

The moment exponential stability is defined in

Definition 0.1. System (0.1) is called exponentially q-stable with respect to the initial data if there are positive numbers c, λ such that all solutions $x(t, b, \varphi)$ $(t \in (-\infty, \infty))$ of the initial value problem $(0.1), (0.1a), (0.1b)$ satisfy the estimate

$$\big(E|x(t, b, \varphi)|^q\big)^{1/q} \le c \exp\{-\lambda t\}\Big(\big(E|b|^q\big)^{1/q} + \operatorname*{ess\,sup}_{\varsigma < 0} \big(E|\varphi(\varsigma)|^q\big)^{1/q}\Big) \;\; (t \ge 0).$$

The next definition is used in the main result of the paper.

Definition 0.2. An invertible matrix $B = (b_{ij})^m_{i,j=1}$ is called inverse-positive if all entries of the matrix B^{-1} are nonnegative.

According to [3], the matrix B will be inverse-positive if $b_{ij} \le 0$ for $i, j = 1, \dots, m$, $i \ne j$ and all diagonal minors of the matrix B are positive. In particular, matrices with strict diagonal dominance and non-positive off-diagonal entriess are inverse-positive.

1 Sufficient stability conditions

As we have already mentioned, we study the moment stability of system (0.1) with respect to the initial data by the W-method, which is based on auxiliary systems. Therefore, along with system (0.1) we consider the following system of linear differential equations with random coefficients:

$$d\widehat{x}(t) = (-B(t)\widehat{x}(t) + f_0(t))\,dt + \sum_{i=1}^{n} f_i(t)\,d\mathcal{B}_i(t) \;\; (t \ge 0), \tag{1.1}$$

where $\widehat{x}=(\widehat{x}_1,\dots,\widehat{x}_n)^T$ is an unknown n-dimensional stochastic process on $(-\infty,\infty)$, $B(t)$ is an $n\times n$-matrix, the entries of which are scalar, progressively measurable stochastic processes on $[0,\infty)$ with the a.s. locally integrable trajectories, while $f_0(t)$, $f_i(t)$ $(i=1,\dots,n)$ are n-dimensional, progressively measurable stochastic processes on $[0,\infty)$ with the a.s. locally square integrable trajectories.

Lemma. *The solutions $\widehat{x}(t)$ of system* (1.1) *can be represented as*

$$\widehat{x}(t)=\widehat{X}(t)\widehat{x}(0)+\int\limits_0^t\widehat{X}(t,\varsigma)f_0(\varsigma)\,d\varsigma+\sum_{i=1}^n\int\limits_0^t\widehat{X}(t,\varsigma)f_i(\varsigma)\,d\mathcal{B}_i(\varsigma)\ \ (t\ge 0),$$

where $\widehat{X}(t,\varsigma)$ $(t\ge 0,\ 0\le\varsigma\le t)$ is the $n\times n$-matrix, the columns of which are solutions of the system $d\widehat{x}(t)=B(t)\widehat{x}(t)\,dt$ $(t\ge 0)$, satisfying $\widehat{X}(t,t)=\overline{E}$ $(t\ge 0)$, while $\widehat{X}(t)\equiv\widehat{X}(t,0)$.

By using the auxiliary system (1.1) and the stated lemma, we can rewrite problem (0.1), (0.1a), (0.1b) in the following equivalent form, where the unknown n-dimensional stochastic process $\overline{x}(t)$ replaces the solution $x(t)$ of system (0.1):

$$\overline{x}(t)=\widehat{X}(t)b+(\Theta(\overline{x}+\overline{\varphi}))(t)\ \ (t\ge 0),$$

where

$$\begin{aligned}(\Theta(\overline{x}+\overline{\varphi}))(t)=&\int\limits_0^t\widehat{X}(t,\varsigma)\Big[B(\varsigma)\overline{x}(\varsigma)-\sum_{j=1}^N A^j(\varsigma)(\overline{x}(h_j(\varsigma))+\overline{\varphi}(h_j(\varsigma)))\Big]\,d\varsigma\\&+\int\limits_0^t\widehat{X}(t,\varsigma)F\Big(\varsigma,\overline{x}(h_1^0(\varsigma))+\overline{\varphi}(h_1^0(\varsigma)),\dots,\overline{x}(h_{m_0}^0(\varsigma))+\overline{\varphi}(h_{m_0}^0(\varsigma))\Big)\,d\varsigma\\&+\sum_{i=1}^m\int\limits_0^t\widehat{X}(t,\varsigma)G^i\Big(\varsigma,\overline{x}(h_1^i(\varsigma))+\overline{\varphi}(h_1^i(\varsigma)),\dots,\overline{x}(h_{m_i}^i(\varsigma))+\overline{\varphi}(h_{m_i}^i(s))\Big)\,d\mathcal{B}_i(\varsigma).\end{aligned}$$

Given $1\le q<\infty$, $\lambda>0$ and a stopping time η we introduce the following vectors:

- $\overline{x}(q,\lambda)=(\overline{x}_1(q,\lambda),\dots,\overline{x}_n(q,\lambda))^T$, where
$$\overline{x}_i(q,\lambda)=\sup_{t\ge 0}\big(E|e^{\lambda t}\overline{x}_i(t)|^q\big)^{1/q},\ \ i=1,\dots,n;$$
- $\overline{x}^{\,\eta}(q,\lambda)=(\overline{x}_1^{\eta}(q,\lambda),\dots,\overline{x}_n^{\eta}(q,\lambda))^T$, where
$$\overline{x}_i^{\eta}(q,\lambda)=\sup_{t\ge 0}\big(E|e^{\lambda t}\overline{x}_i^{\eta}(t)|^q\big)^{1/q},\ \ i=1,\dots,n.$$

Assume that using some auxiliary equation (1.1) we obtain the following estimate:

$$E_n\overline{x}^{\,\eta}(q,\lambda)\le C\overline{x}^{\,\eta}(q,\lambda)+c\Big(\big(E|b|^q\big)^{1/q}+\operatorname*{ess\,sup}_{\varsigma<0}\ \big(E|\varphi(\varsigma)|^q\big)^{1/q}\Big)e_n,\tag{1.2}$$

where C is some nonnegative $n\times n$-matrix, $c\ge 0$, E_n is the identity $n\times n$-matrix, $e_n=(1,\dots,1)^T$ is the n-dimensional vector, and $0\le\eta\le\infty$ is an arbitrary stopping time.

We remind that the stopping time [4] is a random variable $\eta:\Omega\to[0,\infty]$ satisfying the property $\{\omega\in\Omega:\ \eta(\omega)\le t\}\in\mathcal{F}_t$ for any $t\ge 0$, while the "stopped" stochastic process $z^\eta(t)$ is defined by $z^\eta(t)\equiv z(t\wedge\eta)$, where $t\wedge\eta=\min\{t;\eta\}$.

Theorem 1.1. *Assume that $1 \le q < \infty$ and* ***Conditions 1*** *are satisfied. Assume further that estimate* (1.2) *is satisfied for all admissible b, φ and any stopping time $0 \le \eta \le \infty$.*

Then system (0.1) *is exponentially q-stable with respect to the initial data if the matrix $E_n - C$ is inverse-positive.*

To be able to formulate the main result we need

Conditions 2:

- λ is some positive number;
- There exist nonnegative numbers τ_j, $j = 1,\dots,N$, τ_{ij}, $i = 0,\dots,m$, $j = 1,\dots,m_i$ such that $0 \le t - h_j(t) \le \tau_j$, $j = 1,\dots,N$, $0 \le t - h_j^i(t) \le \tau_{ij}$, $i = 0,\dots,m$, $j = 1,\dots,m_i$ $(t \ge 0)$ μ-almost everywhere.
- There exist nonnegative numbers $\overline{F}_{sl}^{j}$, $j = 1,\dots,m_0$, $s,l = 1,\dots,n$ such that
$$\big|F_s(t,u_1,\dots,u_{m_0})\big| \le \sum_{j=1}^{m_0}\sum_{l=1}^{n} \overline{F}_{sl}^{j}|u_j^l|, \quad s = 1,\dots,n, \quad t \ge 0, \quad P \times \mu\text{-almost everywhere.}$$
- There exist nonnegative numbers $\overline{G}_{sl}^{ij}$, $i = 1,\dots,m$, $j = 1,\dots,m_i$, $s,l = 1,\dots,n$ such that
$$\big|G_s^i(t,u_1,\dots,u_{m_i})\big| \le \sum_{j=1}^{m_i}\sum_{l=1}^{n} \overline{G}_{sl}^{ij}|u_j^l|,$$
$$s = 1,\dots,n, \quad i = 1,\dots,m, \quad t \ge 0, \quad P \times \mu\text{-almost everywhere.}$$
- There are subsets $I_s \subset \{1,\dots,N\}$ $(s = 1,\dots,n)$, positive numbers λ_s, $s = 1,\dots,n$ and nonnegative numbers $\overline{a}_{sl}^{j}$, $j = 1,\dots,N$, $s,l = 1,\dots,n$ such that
$$\sum_{j\in I_s} a_{ss}^j(t) \ge \lambda_s, \quad s = 1,\dots,n,$$
$$|a_{sl}^j(t)| \le \overline{a}_{sl}^{j}, \quad j = 1,\dots,N, \quad s,l = 1,\dots,n, \quad t \ge 0, \quad P \times \mu\text{-almost everywhere.}$$

Stability conditions will be formulated in terms of the special $n \times n$-matrix C, whose entries are defined as follows:

$$c_{ss} = \frac{1}{\lambda_s}\left[\sum_{j\in I_s}\overline{a}_{ss}^{j}\tau_j\Big(\sum_{j=1}^{N}\overline{a}_{ss}^{j} + \overline{F}_{ss} + \frac{c_p}{\sqrt{\tau_j}}\,\overline{G}_{ss}\Big) + \sum_{j=1, j\notin I_s}^{N}\overline{a}_{ss}^{j} + \overline{F}_{ss}\right] + \frac{c_p}{\sqrt{2\lambda_s}}\,\overline{G}_{ss}, \quad s = 1,\dots,n,$$
$$c_{sl} = \frac{1}{\lambda_s}\left[\sum_{j\in I_s}\overline{a}_{ss}^{j}\tau_j\Big(\sum_{j=1}^{N}\overline{a}_{sl}^{j} + \overline{F}_{sl} + \frac{c_p}{\sqrt{\tau_j}}\,\overline{G}_{sl}\Big) + \sum_{j=1}^{N}\overline{a}_{sl}^{j} + \overline{F}_{sl}\right] + \frac{c_p}{\sqrt{2\lambda_s}}\,\overline{G}_{sl}, \quad s,l = 1,\dots,n, \quad s \ne l,$$

where
$$\overline{F}_{sl} = \sum_{j=1}^{m_0}\overline{F}_{sl}^{j}, \quad \overline{G}_{sl} = \sum_{i=1}^{m}\sum_{j=1}^{m_i}\overline{G}_{sl}^{ij}, \quad s,l = 1,\dots,n.$$

Here the constant c_p comes from the estimate

$$\left(E\left|\int_0^t f(\varsigma)\,d\mathcal{B}(\varsigma)\right|^{2p}\right)^{1/(2p)} \le c_p\left(E\left(\int_0^t |f(\varsigma)|^2 d\varsigma\right)^{p}\right)^{1/(2p)}, \tag{1.3}$$

where $f(\varsigma)$ is an arbitrary scalar, progressive measurable stochastic process and $\mathcal{B}(\varsigma)$ is the scalar Wiener process. Estimate (1.3) follows from the inequality mentioned in [4, p. 65], where the expressions for c_p can be found as well.

Theorem 1.2. *Let $1 \le p < \infty$ and* ***Conditions 1-2*** *be satisfied. If the matrix $E_n - C$ is inverse positive, then system* (0.1) *is exponentially* $2p$*-stable with respect to initial data for any* $0 < \lambda < \min\{\lambda_s,\ s = 1, \dots, n\}$.

2 An example

Let us fix a number $1 \le p < \infty$ and consider the system of nonlinear Itô equations

$$dx(t) = -\sum_{j=1}^{N} A^j x(t-h_j)\,dt + \sum_{j=1}^{m_0} A^{0j} x^{\alpha_j^0}(t-h_j^0)\,dt + \sum_{i=1}^{m}\sum_{j=1}^{m_i} A^{ij} x^{\alpha_j^i}(t-h_j^i)\,d\mathcal{B}_i(t) \ \ (t \ge 0), \quad (2.1)$$

where $A^j = (a_{sl}^j)_{s,l=1}^n$, $j = 1, \dots, N$, $A^{ij} = (a_{sl}^{ij})_{s,l=1}^n$, $i = 0, \dots, m$, $j = 1, \dots, m_i$ are real $n \times n$-matrices, $h_j \ge 0$, $j = 1, \dots, N$, $h_j^i \ge 0$, $i = 0, \dots, m$, $j = 1, \dots, m_i$ are real numbers, and α_j^i, $i = 0, \dots, m$, $j = 1, \dots, m_i$ are real numbers satisfying the inequalities $0 < \alpha_j^i \le 1$, $i = 0, \dots, m$, $j = 1, \dots, m_i$.

Assume that

$$\sum_{j=1}^{N} a_{ss}^j = \lambda_s > 0, \quad s = 1, \dots, n$$

and

$$\overline{F}_{sl} = \sum_{j=1}^{m_0} |a_{sl}^{0j}|, \quad \overline{G}_{sl} = \sum_{i=1}^{m}\sum_{j=1}^{m_i} |a_{sl}^{ij}|, \quad s, l = 1, \dots, n,$$

and the $n \times n$-matrix $E_n - C$ is inverse-positive, where C consists of the following entries:

$$c_{sl} = \frac{1}{\lambda_s}\bigg[\sum_{j=1}^{N} |a_{ss}^j| h_j \Big(\sum_{j=1}^{N} |a_{sl}^j| + \overline{F}_{sl} + \frac{c_p}{\sqrt{h_j}}\,\overline{G}_{sl}\Big) + \sum_{j=1}^{N} |a_{sl}^j| + \overline{F}_{sl}\bigg] - \frac{c_p}{\sqrt{\lambda_s}}\,\overline{G}_{sl}, \quad s, l = 1, \dots, n.$$

Then from Theorem 1.2 it follows that the nonlinear system (2.1) is exponentially $2p$-stable with respect to the initial data.

References

[1] N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations: Methods and Applications.* Contemporary Mathematics and Its Applications, 3. Hindawi Publishing Corporation, Cairo, 2007.

[2] N. V. Azbelev and P. M. Simonov, *Stability of Differential Equations with Aftereffect.* Stability and Control: Theory, Methods and Applications, 20. Taylor & Francis, London, 2003.

[3] R. Bellman, *Introduction to Matrix Analysis.* (Russian) Second edition McGraw-Hill Book Co., New York–Düsseldorf–London, 1970.

[4] R. Sh. Liptser and A. N. Shiryayev, *Theory of Martingales.* Mathematics and its Applications (Soviet Series), 49. Kluwer Academic Publishers Group, Dordrecht, 1989.