Investigation and Approximate Solution of One Nonlinear Degenerate Integro-Differential Equation of Parabolic Type

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Many natural processes in mathematical modeling can be described by the initial-boundary value problems posed for nonlinear parabolic differential and integro-differential models (see, for example, [3, 4, 8, 10, 13, 14, 16–18] and the references therein). Investigation and construction of algorithms for approximate solutions to these problems are the actual sphere of contemporary mathematical physics and numerical analysis.

A lot of scientific works are dedicated to the investigation and numerical resolution of integro-differential models (see, for example, [3, 8, 10, 14, 16, 18] and the references therein).

One type of integro-differential nonlinear parabolic model is obtained at the mathematical simulation of processes of electromagnetic field penetration in the substance. Based on the Maxwell system [12], this model at first appeared and was studied in [5]. Based on the works [1, 2, 5], the models of such type are investigated in many works (see, for example, [6, 7, 9, 11, 15] and the references therein). Equations and systems of such types still yield to the investigation for special cases. In this direction, the latest and rather complete bibliography can be found in the following monographs [8, 10].

Many scientific papers are devoted to the construction and investigation of discrete analogs of the above-mentioned integro-differential models and for problems similar to them. There are still many open questions in this direction.

The present work is dedicated to the investigation and approximate resolution of the initial-boundary value problem for the following equation

$$\frac{\partial U}{\partial t} + A_1U + A_2U + A_3U = f(x,t),$$

where

$$A_1U = -\frac{\partial}{\partial x}\left\{\left[\int_0^t \Big(\frac{\partial U}{\partial x}\Big)^2\,d\tau\right]\frac{\partial U}{\partial x}\right\},$$

$$A_2U = -\left[\int_0^1\int_0^t \Big(\frac{\partial U}{\partial x}\Big)^2\,d\tau\,dx\right]\frac{\partial^2 U}{\partial x^2}\,,$$

$$A_3U = -\frac{\partial}{\partial x}\left[\Big(\frac{\partial U}{\partial x}\Big)^2\frac{\partial U}{\partial x}\right].$$

The purpose of this note is to analyze such type of degenerate equation. In [9] unique solvability and convergence of the semi-discrete scheme with respect to the spatial derivative and finite difference scheme for $\partial U/\partial t + A_1U + A_3U = f(x,t)$ equation are studied. The present work is dedicated to studying such questions for $\partial U/\partial t + A_2U + A_3U = f(x,t)$.

So, the investigated problem has the following form. In the rectangle $Q = (0,1) \times (0,T)$ where T is a fixed positive constant, we consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \left[\int_0^1 \int_0^t \left(\frac{\partial U}{\partial x} \right)^2 d\tau \, dx \right] \frac{\partial^2 U}{\partial x^2} - \frac{\partial}{\partial x} \left[\left(\frac{\partial U}{\partial x} \right)^2 \frac{\partial U}{\partial x} \right] = f(x,t), \tag{1}$$

$$(0,t) = U(1,t) = 0, \quad t \in [0,T], \tag{2}$$

$$U(x,0) = U_0(x), \quad x \in [0,1]. \tag{3}$$

Here $f = f(x,t)$, $U_0 = U_0(x)$ are given functions of their arguments and $U = U(x,t)$ is an unknown function. It is necessary to mention that (1) is a degenerate type parabolic equation with integro-differential and p-Laplacian term ($p = 4$).

Using one modification of the compactness method developed in [17] (see also [16]) the following uniqueness and existence statement takes place.

Theorem 1. *If $f \in W_2^1(Q)$, $f(x,0) = 0$, $U_0, V_0 \in \overset{\circ}{W}{}_2^1(0,1)$, then there exists the unique solution U of problem (1)–(3) satisfying the following properties:*

$$U \in L_4(0,T; \overset{\circ}{W}{}_4^1(0,1) \cap W_2^2(0,1)), \quad \frac{\partial U}{\partial t} \in L_2(Q), \quad \sqrt{T-t}\, \frac{\partial^2 U}{\partial t \partial x} \in L_2(Q).$$

Here usual well-known spaces are used.

In order to describe the space-discretization for problem (1)–(3), let us introduce nets:

$$\omega_h = \{x_i = ih, \ i = 1,2,\dots,M-1\}, \quad \overline{\omega}_h = \{x_i = ih, \ i = 0,1,\dots,M\}$$

with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i,t) is designed by $u_i = u_i(t)$. The exact solution of problem (1)–(3) at (x_i,t) is denoted by $U_i = U_i(t)$ and is assumed to exist and be smooth enough.

Approximating the space derivatives by the differences:

$$u_{x,i} = \frac{u_{i+1} - u_i}{h}, \quad u_{\overline{x},i} = \frac{u_i - u_{i-1}}{h}, \quad u_{\overline{x}x,i} = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},$$

let us correspond to problem (1)–(3) the following semi-discrete scheme:

$$\frac{du_i}{dt} - h \sum_{l=1}^{M} \int_0^t (u_{\overline{x},l})^2 d\tau \, u_{\overline{x}x,i} - \left[(u_{\overline{x},i})^2 u_{\overline{x},i} \right]_{x,i} = f(x_i,t), \quad i = 1,2,\dots,M-1, \tag{4}$$

$$u_0(t) = u_M(t) = 0, \quad t \in [0,T], \tag{5}$$

$$u_i(0) = U_{0,i}, \quad i = 0,1,\dots,M, \tag{6}$$

which approximates problem (1)–(3) on smooth solutions with the first order of accuracy with respect to h.

Problem (4)–(6) is a Cauchy problem for a nonlinear system of ordinary integro-differential equations. It is not difficult to obtain the following estimate for (4)–(6)

$$\|u\|_h^2 + \int_0^t \|u_{\overline{x}}]|_h^2 \, d\tau < C,$$

where

$$\|u\|_h^2 = (u,u)_h, \quad (u,v)_h = \sum_{i=1}^{M-1} u_i v_i h, \quad \|u_{\overline{x}}]|_h^2 = (u_{\overline{x}}, u_{\overline{x}}]_h, \quad (u_{\overline{x}}, v_{\overline{x}}]_h = \sum_{i=1}^{M} u_{\overline{x},i} v_{\overline{x},i} h.$$

So, the semi-discrete scheme (4)–(6) is stable with respect to initial data and the right-hand side of equation (4).

Here and below in Theorem 2 by C a generic positive constant independent of the mesh parameter h is denoted. This estimate gives us the global existence of a solution to problem (4)–(6).

Using an approach of the work [7], here in Theorem 2 and below in Theorem 3 for the investigation of the finite-difference scheme, the convergence of the approximate solutions is proved.

The following statement takes place.

Theorem 2. *If problem* (1)–(3) *has a sufficiently smooth solution* $U = U(x,t)$, *then the solution* $u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$ *of the semi-discrete scheme* (4)–(6) *tends to the solution* $U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$ *as* $h \to 0$ *and the following estimate is true*

$$\|u(t) - U(t)\|_h \le Ch.$$

In order to describe the fully discrete analog of problem (1)–(3), let us construct a grid on the rectangle $\overline{Q}$. For using the time-discretization in equation (1), the net is introduced as follows $\omega_\tau = \{t_j = j\tau,\ j = 0,1,\dots,J\}$, with $\tau = T/J$ and $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $u_i^j = u(x_i, t_j)$.

Let us correspond to problem (1)–(3) the following implicit finite difference scheme:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \Big[\tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (u_l^k)^2\Big] u_{\overline{x}x,i}^{j+1} - \Big\{\big[(u_{\overline{x},i}^{j+1})^2\big] u_{\overline{x},i}^{j+1}\Big\}_{x,i} = f_i^{j+1}, \tag{7}$$

$$i = 1,2,\dots,M-1, \quad j = 0,1,\dots,J-1,$$

$$u_0^j = u_M^j = 0, \quad j = 0,1,\dots,J, \tag{8}$$

$$u_i^0 = U_{0,i}, \quad i = 0,1,\dots,M. \tag{9}$$

So, the system of nonlinear algebraic equations (7)–(9) is obtained, which approximates problem (1)–(3) on the sufficiently smooth solution by the order $O(\tau + h)$.

As for the semi-discrete scheme (4)–(6), we easily obtain the estimate

$$\max_{0 \le j\tau \le T} \|u^j\|_h^2 + \sum_{k=1}^{J} \|u_{\overline{x}}^k]|_h^2 \tau < C,$$

which guarantees the stability and solvability of scheme (7)–(9). It is proved also that system (7)–(9) has a unique solution. Here and below C is a positive constant independent from time and spatial steps τ and h.

The following main conclusion is valid for scheme (7)–(9).

Theorem 3. *If problem* (1)–(3) *has a sufficiently smooth solution* $U = U(x,t)$, *then the solution* $u^j = (u_1^j, u_2^j, \dots, u_{M-1}^j)$, $j = 1,2,\dots,J$ *of the difference scheme* (7)–(9) *tends to the solution* $U^j = (U_1^j, U_2^j, \dots, U_{M-1}^j)$, $j = 1,2,\dots,J$ *as* $\tau \to 0$, $h \to 0$ *and the following estimate is true*

$$\|u^j - U^j\|_h \le C(\tau + h), \quad j = 1,2,\dots,J.$$

Note that for solving the difference scheme (7)–(9) Newton's iterative process is used and various numerical experiments are done. These experiments agree with theoretical research.

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