Periodic-Type Solutions for Differential Equations with Positively Homogeneous Operators

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1 Introduction

In the paper [1], we show how a rather abstract Fredholm-type result from [2] can be successfully applied to study ω-periodic solutions to the following second order differential equation with a $(\lambda+1)$-Laplacian and maxima:

$$\big(|u'(t)|^{\lambda}\operatorname{sgn}u'(t)\big)' = g(t)\max\big\{|u(s)|^{\lambda}\operatorname{sgn}u(s):\ \mu(t)\le s\le\tau(t)\big\} + f_0(t), \tag{1.1}$$

where $f_0, g \in L([0,\omega];\mathbb{R})$, $\lambda > 0$, and $\mu, \tau : [0,\omega] \to [0,\omega]$ are measurable functions satisfying $\mu(t) \le \tau(t)$ for almost all t belonging to the period segment $[0,\omega]$. Two of our main results stated in Section 2, Corollaries 2.3 and 2.4, present easily verifiable conditions for the existence of at least one ω-periodic solution to the equation (1.1) for each perturbation $f_0(t)$. Importantly, the leading coefficient $g(t)$ in (1.1) can oscillate: in such a case, we will assume that either positive or negative part of $g(t)$ dominates the part of $g(t)$ having the opposite sign, see Corollaries 2.3 and 2.4 for the precise formulations. Note that the uniqueness of periodic solutions is not analysed in the present work. Nevertheless, it is known that even the first order periodic equation with the right-hand side as in (1.1) and constant coefficient $g(t)$ can have multiple (or even infinite number of) subharmonic periodic solutions for a class of sine-like forcing terms $f_0(t)$. We leave the aforementioned uniqueness problem for equation (1.1) as an interesting open question.

Now, our approach allows to consider more general objects in the form of two-dimensional system of functional differential equations

$$u_1'(t) = f_1(u_1,u_2)(t), \tag{1.2}$$

$$u_2'(t) = f_2(u_1,u_2)(t),\ \ t\in[0,\omega], \tag{1.3}$$

subjected to the periodic-type boundary value conditions

$$u_1(\omega)-u_1(0) = h_1(u_1,u_2),\quad u_2(\omega)-u_2(0) = h_2(u_1,u_2). \tag{1.4}$$

Here $f_i : C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ $(i = 1,2)$ are continuous operators satisfying Carathéodory conditions, i.e., for every $r > 0$ there exists $q_r \in L([0,\omega];\mathbb{R}_+)$ such that

$$|f_1(u_1,u_2)(t)| + |f_2(u_1,u_2)(t)| \le q_r(t) \ \text{ for a.e. } \ t \in [0,\omega] \ \text{ whenever } \ \|u_1\|_C + \|u_2\|_C \le r,$$

and $h_i : C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R}) \to \mathbb{R}$ $(i = 1,2)$ are continuous functionals bounded on every ball by a constant, i.e., for every $r > 0$ there exists $M_r > 0$ such that

$$|h_1(u_1,u_2)| + |h_2(u_1,u_2)| \le M_r \ \text{ whenever } \ \|u_1\|_C + \|u_2\|_C \le r.$$

By a solution to the system $(1.2), (1.3)$ we understand a vector-valued function $(u_1,u_2) \in C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R})$ with absolutely continuous components that satisfy the equalities (1.2) and (1.3) almost everywhere in $[0,\omega]$. By a solution to the problem (1.2)–(1.4) we understand a solution to $(1.2), (1.3)$ which satisfies (1.4).

Before presenting our main results in Section 2, let us introduce basic notation used in this work:

$\mathbb{R}$ is a set of all real numbers;

$C([0,\omega];\mathbb{R})$ is a Banach space of continuous functions $u : [0,\omega] \to \mathbb{R}$ endowed with the norm

$$\|u\|_C = \max\big\{|u(t)| : \ t \in [0,\omega]\big\};$$

$L([0,\omega];\mathbb{R})$ is a Banach space of Lebesgue integrable functions $u : [0,\omega] \to \mathbb{R}$ endowed with the norm

$$\|u\|_L = \int_0^\omega |u(t)|\, dt;$$

if $g \in L([0,\omega];\mathbb{R})$ then $[g]_+$, resp. $[g]_-$, denotes the non-negative, resp. nonpositive, part of the function g, i.e.,

$$[g]_+(t) \overset{def}{=} \frac{|g(t)| + g(t)}{2}, \quad [g]_-(t) \overset{def}{=} \frac{|g(t)| - g(t)}{2} \ \text{ for a.e. } \ t \in [0,\omega];$$

$\mathscr{P}(\lambda)$, where $\lambda > 0$, is a set of all continuous nondecreasing operators $p : C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ satisfying Carathéodory conditions which are positively homogeneous with a degree λ, i.e., for every $c > 0$ and $u \in C([0,\omega];\mathbb{R})$ the following identity holds:

$$p(cu)(t) = c^\lambda p(u)(t) \ \text{ for a.e. } \ t \in [0,\omega].$$

Let $\mu, \tau : [0,\omega] \to [0,\omega]$ be measurable functions. Then, for every $t \in [0,\omega]$, we put $I(\mu(t),\tau(t)) = [\mu(t),\tau(t)]$ if $\mu(t) \le \tau(t)$ and $I(\mu(t),\tau(t)) = \varnothing$ otherwise.

$\mathscr{S}$ is a set of all mappings $S : [0,\omega] \to 2^{[0,\omega]}$ such that $S(t)$ is a union of at most countable number of intervals $(\mu_k(t),\tau_k(t))$, where $\mu_k, \tau_k : [0,\omega] \to [0,\omega]$ are measurable functions satisfying $\mu_k(t) \le \tau_k(t)$ for almost all $t \in [0,\omega]$.

Note that the function $t \mapsto \sup\{|u(s)|^\lambda \operatorname{sgn} u(s) : \ s \in S(t)\}$ is measurable whenever $u \in C([0,\omega];\mathbb{R})$, $S \in \mathscr{S}$, and $\lambda > 0$ (we put $\sup\varnothing = -\infty$).

For given $p \in \mathscr{P}(\lambda)$ and a number $\delta \in [0,1]$ we define the operator $p(\,\cdot\,;\delta) : C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ and a non-negative numbers $\widehat{P}(\delta)$ and $P(\delta)$ in the following way:

$$p(u;\delta)(t) \overset{def}{=} (1-\delta)p(u)(t) - \delta p(-u)(t) \ \text{ for a.e. } \ t \in [0,\omega], \quad \widehat{P}(\delta) \overset{def}{=} \int_0^\omega p(1;\delta)(t)\, dt,$$

$$P(\delta) \overset{def}{=} \max\left\{ \int_x^y p(1;\delta)(t)\, dt + \int_y^{x+\omega} p(1;1-\delta)(t)\, dt : \ x \in [0,\omega], \ y \in [x, x+\omega] \right\},$$

where
$$p(1;\nu)(t)=p(1;\nu)(t-\omega) \ \text{ for a.e. } t\in(\omega,2\omega], \quad \nu=\delta,1-\delta.$$

Obviously, $\widehat{P}(\delta)\le P(\delta)$ and $-p(-u;\delta)\equiv p(u;1-\delta)$ for every $u\in C([0,\omega];\mathbb{R})$ and $\delta\in[0,1]$. It can be also easily verified that
$$P(\delta)=P(1-\delta) \ \text{ for } \ \delta\in[0,1]. \tag{1.5}$$

Furthermore, for given $p_0\in\mathscr{P}(\lambda_1)$ and $p_1,p_2\in\mathscr{P}(\lambda_2)$ we define the following functions:
$$q_1(t,\rho)\overset{def}{=}\sup\Big\{|f_1(u_1,u_2)(t)-p_0(u_2)(t)|: \ \|u_1\|_C\le\rho, \ \|u_2\|_C\le\rho^{\lambda_2}\Big\} \tag{1.6}$$
$$\text{for a.e. } t\in[0,\omega],$$
$$q_2(t,\rho)\overset{def}{=}\sup\Big\{|f_2(u_1,u_2)(t)-p_1(u_1)(t)+p_2(u_1)(t)|: \ \|u_1\|_C\le\rho^{\lambda_1}, \ \|u_2\|_C\le\rho\Big\} \tag{1.7}$$
$$\text{for a.e. } t\in[0,\omega],$$
$$\eta_k(\rho)\overset{def}{=}\sup\Big\{|h_k(u_1,u_2)|: \ \|u_k\|_C\le\rho, \ \|u_{3-k}\|_C\le\rho^{\lambda_{3-k}}\Big\} \ (k=1,2). \tag{1.8}$$

2 Main results

Now we can formulate our main results. The proofs of the results slightly differ depending on the values of λ_i. Therefore it is convenient formulate assertions for two separate cases. Thus, Theorem 2.1 deals with the case when $\lambda_2\ge1$, Theorem 2.2 can be applied in the case when $\lambda_2<1$.

Theorem 2.1. *Let $\lambda_1,\lambda_2>0$, $\lambda_1\lambda_2=1$, and let there exist $p_0\in\mathscr{P}(\lambda_1)$ and $p_1,p_2\in\mathscr{P}(\lambda_2)$ such that*
$$\lim_{\rho\to+\infty}\int_0^\omega\frac{q_k(s,\rho)}{\rho}\,ds=0, \qquad \lim_{\rho\to+\infty}\frac{\eta_k(\rho)}{\rho}=0 \ \ (k=1,2), \tag{2.1}$$
where q_k and η_k are given by (1.6)–(1.8). *Let, moreover, $\lambda_2\ge1$, $p_0(1)\not\equiv0$, $p_0(-1)\not\equiv0$, and let there exist $i\in\{1,2\}$ such that, for every $\delta\in[0,1]$, the following inequalities hold:*
$$\frac{P_0(\delta)}{2^{1+\lambda_1}}P_i^{\lambda_1}(\delta)<1, \quad \widehat{P}_i^{\lambda_1}(\delta)<\Big(1-\frac{P_0(\delta)}{2^{1+\lambda_1}}\,\widehat{P}_i^{\lambda_1}(\delta)\Big)\widehat{P}_{3-i}^{\lambda_1}(\delta), \tag{2.2}$$
$$\frac{P_0^{\lambda_2}(\delta)}{2^{2+\lambda_2}}\,P_{3-i}(\delta)<2^{\lambda_2}-1+\sqrt{1-\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\,P_i(\delta)}\,. \tag{2.3}$$

Then the problem (1.2)–(1.4) *has at least one solution.*

Theorem 2.2. *Let $\lambda_1,\lambda_2>0$, $\lambda_1\lambda_2=1$, and let there exist $p_0\in\mathscr{P}(\lambda_1)$ and $p_1,p_2\in\mathscr{P}(\lambda_2)$ such that* (2.1) *is fulfilled where q_k and η_k are given by* (1.6)–(1.8). *Let, moreover, $\lambda_2<1$, $p_0(1)\not\equiv0$, $p_0(-1)\not\equiv0$, and let there exist $i\in\{1,2\}$ such that, for every $\delta\in[0,1]$, the following inequalities hold:*
$$\frac{P_0(\delta)}{4}\,P_i^{\lambda_1}(\delta)<1, \quad \widehat{P}_i^{\lambda_1}(\delta)<\Big(1-\frac{P_0(\delta)}{2^{1+\lambda_1}}\,\widehat{P}_i^{\lambda_1}(\delta)\Big)\widehat{P}_{3-i}^{\lambda_1}(\delta), \tag{2.4}$$
$$\frac{P_0^{\lambda_2}(\delta)}{2^{2\lambda_2+1}}\,P_{3-i}(\delta)<1+\sqrt{1-\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}}\,P_i(\delta)}\,. \tag{2.5}$$

Then the problem (1.2)–(1.4) *has at least one solution.*

In the case when the operator $p \in \mathscr{P}(\lambda)$ is homogeneous on the constant functions, i.e., if $p(-1) \equiv -p(1)$, then the numbers $\widehat{P}(\delta)$, $P(\delta)$ take more simple form. More precisely, they do not depend on δ anymore and

$$\widehat{P}(\delta) = P(\delta) = \int_0^\omega p(1)(t)\,dt.$$

The typical operator having the above-described property is an operator defined by means of suprema of the function u over certain subsets of its domain:

$$p(u)(t) \stackrel{def}{=} g(t) \sup\big\{|u(s)|^\lambda \operatorname{sgn} u(s): \ \ s \in S(t)\big\},$$

where $g \in L([0,\omega];\mathbb{R})$ and $S \in \mathscr{S}$. Therefore, considering the system

$$u_1'(t) = g_0(t) \sup\big\{|u_2(s)|^{\lambda_1} \operatorname{sgn} u_2(s): \ \ s \in S_0(t)\big\} + \widetilde{f}_1(u_1,u_2)(t), \tag{2.6}$$

$$\begin{aligned} u_2'(t) = {}& g_1(t) \sup\big\{|u_1(s)|^{\lambda_2} \operatorname{sgn} u_1(s): \ \ s \in S_1(t)\big\} \\ & - g_2(t) \sup\big\{|u_1(s)|^{\lambda_2} \operatorname{sgn} u_1(s): \ \ s \in S_2(t)\big\} + \widetilde{f}_2(u_1,u_2)(t), \end{aligned} \tag{2.7}$$

where $g_i \in L([0,\omega];\mathbb{R}_+)$, $S_i \in \mathscr{S}$ $(i=0,1,2)$, and $\widetilde{f}_1, \widetilde{f}_2 : C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ are continuous operators satisfying Carathéodory conditions, from Theorems 2.1 and 2.2 we derive the following assertions:

Corollary 2.1. *Let $\lambda_1, \lambda_2 > 0$, $\lambda_1\lambda_2 = 1$, and let* (2.1) *be fulfilled where*

$$q_k(t,\rho) \stackrel{def}{=} \sup\Big\{|\widetilde{f}_k(u_1,u_2)(t)|: \ \ \|u_k\|_C \le \rho, \ \ \|u_{3-k}\|_C \le \rho^{\lambda_{3-k}}\Big\} \ \text{for a.e. } t \in [0,\omega] \tag{2.8}$$

and η_k are given by (1.8). *Let, moreover, $\lambda_2 \ge 1$ and $g_i(t) \ge 0$ $(i=0,1,2)$ for almost every $t \in [0,\omega]$, $g_0 \not\equiv 0$, and let there exist $i \in \{1,2\}$ such that the following inequalities hold:*

$$\frac{\|g_0\|_L}{2^{1+\lambda_1}}\,\|g_i\|_L^{\lambda_1} < 1, \quad \|g_i\|_L^{\lambda_1} < \Big(1 - \frac{\|g_0\|_L}{2^{1+\lambda_1}}\,\|g_i\|_L^{\lambda_1}\Big)\|g_{3-i}\|_L^{\lambda_1},$$

$$\frac{\|g_0\|_L^{\lambda_2}}{2^{2+\lambda_2}}\,\|g_{3-i}\|_L < 2^{\lambda_2} - 1 + \sqrt{1 - \frac{\|g_0\|_L^{\lambda_2}}{2^{1+\lambda_2}}\,\|g_i\|_L}\,.$$

Then the problem (2.6), (2.7), (1.4) *has at least one solution.*

Corollary 2.2. *Let $\lambda_1, \lambda_2 > 0$, $\lambda_1\lambda_2 = 1$, and let* (2.1) *be fulfilled where q_k and η_k are given by* (2.8) *and* (1.8), *respectively. Let, moreover, $\lambda_2 < 1$ and $g_i(t) \ge 0$ $(i=0,1,2)$ for almost every $t \in [0,\omega]$, $g_0 \not\equiv 0$, and let there exist $i \in \{1,2\}$ such that the following inequalities hold:*

$$\frac{\|g_0\|_L}{4}\,\|g_i\|_L^{\lambda_1} < 1, \quad \|g_i\|_L^{\lambda_1} < \Big(1 - \frac{\|g_0\|_L}{2^{1+\lambda_1}}\,\|g_i\|_L^{\lambda_1}\Big)\|g_{3-i}\|_L^{\lambda_1},$$

$$\frac{\|g_0\|_L^{\lambda_2}}{2^{2\lambda_2+1}}\,\|g_{3-i}\|_L < 1 + \sqrt{1 - \frac{\|g_0\|_L^{\lambda_2}}{4^{\lambda_2}}\,\|g_i\|_L}\,.$$

Then the problem (2.6), (2.7), (1.4) *has at least one solution.*

Now, consider the particular case of equation (1.1) where $f_0, g \in L([0,\omega];\mathbb{R})$, $\lambda > 0$, and $\mu, \tau : [0,\omega] \to [0,\omega]$ are measurable functions satisfying $\mu(t) \le \tau(t)$ for almost all $t \in [0,\omega]$. Obviously, in such a case, we can invoke our previous results setting $g_0 \equiv 1$, $g_1 \equiv [g]_+$, $g_2 \equiv [g]_-$, $\lambda_1 = 1/\lambda$, $\lambda_2 = \lambda$, and $S_0(t) = \{t\}$, $S_1(t) = S_2(t) = [\mu(t), \tau(t)]$ for almost all $t \in [0,\omega]$. Thus, Corollaries 2.1 and 2.2 yields the following assertions dealing with the equation (1.1).

Corollary 2.3. *Let* $\lambda \geq 1$ *and let there exist* $\sigma \in \{-1, 1\}$ *such that*

$$\big\|[\sigma g]_+\big\|_L < \frac{2^{1+\lambda}}{\omega^\lambda}\,,$$

$$\frac{\|[\sigma g]_+\|_L}{(1-\omega 2^{1+1/\lambda}\|[\sigma g]_+\|_L^{1/\lambda})^\lambda} < \|[\sigma g]_-\|_L < \frac{2^{2+\lambda}}{\omega^\lambda}\bigg(2^\lambda - 1 + \sqrt{1-\frac{\omega^\lambda}{2^{1+\lambda}}\,\big\|[\sigma g]_+\big\|_L}\,\bigg).$$

Then the equation (1.1) *has at least one solution* u *that satisfies* $u(0) = u(\omega)$, $u'(0) = u'(\omega)$.

Corollary 2.4. *Let* $0 < \lambda < 1$ *and let there exist* $\sigma \in \{-1, 1\}$ *such that*

$$\big\|[\sigma g]_+\big\|_L < \Big(\frac{4}{\omega}\Big)^\lambda,$$

$$\frac{\|[\sigma g]_+\|_L}{(1-\frac{\omega}{2^{1+1/\lambda}}\|[\sigma g]_+\|_L^{1/\lambda})^\lambda} < \big\|[\sigma g]_-\big\|_L < \frac{2^{2\lambda+1}}{\omega^\lambda}\bigg(1 + \sqrt{1-\Big(\frac{\omega}{4}\Big)^\lambda\big\|[\sigma g]_+\big\|_L}\,\bigg).$$

Then the equation (1.1) *has at least one solution* u *that satisfies* $u(0) = u(\omega)$, $u'(0) = u'(\omega)$.

References

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