On a Lower Estimate for the First Eigenvalue of a Sturm–Liouville Problem

S. Ezhak, M. Telnova

Plekhanov Russian University of Economics, Moscow, Russia
E-mails: svetlana.ezhak@gmail.com; mytelnova@yandex.ru

1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all locally integrable on $(0,1)$ functions with non-negative values such that the following integral conditions hold:

$$\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx = 1, \quad \gamma \neq 0, \tag{3}$$

$$\int_0^1 x(1-x)Q(x)\,dx < \infty. \tag{4}$$

A function y is a *solution* of problem (1), (2) if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

In Theorem 1 [2], it was proved that if condition (4) does not hold, then for any $0 \leq p \leq \infty$, there is no non–trivial solution y of equation (1) with properties $y(0) = 0$, $y'(0) = p$.

If $\gamma < 0$, $\alpha \leq 2\gamma - 1$ or $\beta \leq 2\gamma - 1$, then the set $T_{\alpha,\beta,\gamma}$ is empty; for other values α, β, γ, $\gamma \neq 0$, the set $T_{\alpha,\beta,\gamma}$ is not empty [4, Chapter 1, § 2, Theorem 3]. Since for $\gamma < 0$, $\alpha \leq 2\gamma-1$ or $\beta \leq 2\gamma-1$ there is no function Q satisfying (3) and (4) taken together, then problem (1)–(4) is not considered for these parameters.

This work gives estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

Consider the functional

$$R[Q,y] = \frac{\int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}.$$

If condition (4) is satisfied, then the functional $R[Q,y]$ is bounded below in $H_0^1(0,1)$ [3]. It was proved [2,3] that for any $Q \in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y].$$

For any $Q \in T_{\alpha,\beta,\gamma}$, we have

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y] \leq \inf_{y \in H_0^1(0,1)\setminus\{0\}} \frac{\int\limits_0^1 y'^2\,dx}{\int\limits_0^1 y^2\,dx} = \pi^2.$$

2 Main results

Theorem 2.1. *If $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$ such that $m_{\alpha,\beta,\gamma} = R[Q_*, u]$. Moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{\gamma+1}{\gamma-1}} \tag{5}$$

and the integral condition

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{2\gamma}{\gamma-1}}\,dx = 1. \tag{6}$$

Theorem 2.2.

(1) *If $\gamma = 1$, $\alpha, \beta \leqslant 0$, then $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.*

(2) *If $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ or $\alpha \leqslant 0 < \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(3) *If $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(4) *If $\gamma > 1$, $\alpha, \beta \leqslant \gamma$, then $m_{\alpha,\beta,\gamma} = 0$.*

(5) *If $\gamma > 1$, $\gamma < \alpha \leqslant 2\gamma - 1$ or $\gamma < \beta \leqslant 2\gamma - 1$, then $m_{\alpha,\beta,\gamma} \leqslant 0$.*

(6) *If $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$, $0 < \gamma < 1$, $-\infty < \alpha, \beta < +\infty$ or if $\gamma \geqslant 1$, $\alpha > 2\gamma - 1$ or $\beta > 2\gamma - 1$, then $m_{\alpha,\beta,\gamma} = -\infty$.*

Let us show that if $\gamma \geqslant 1$, $\alpha > 2\gamma - 1$, $-\infty < \beta < \infty$, then we have $m_{\alpha,\beta,\gamma} = -\infty$ (the case $\gamma \geqslant 1$, $\beta > 2\gamma - 1$, $-\infty < \beta < \infty$ is similar).

Consider the functions $Q_\varepsilon \in T_{\alpha,\beta,\gamma}$ and $y_0 \in H_0^1(0,1)$:

$$Q_\varepsilon(x) = \begin{cases} (\alpha+1)^{\frac{1}{\gamma}} \varepsilon^{-\frac{\alpha+1}{\gamma}} (1-x)^{-\frac{\beta}{\gamma}}, & x \in [0, \varepsilon], \\ 0, & x \in (\varepsilon, 1], \end{cases}$$

$$y_0(x) = \begin{cases} x^\theta, & x \in \Big[0, \dfrac{1}{2}\Big], \\ (1-x)^\theta, & x \in \Big(\dfrac{1}{2}, 1\Big], \quad \theta > \dfrac{1}{2}\,. \end{cases}$$

We have

$$\int\limits_0^1 Q_\varepsilon(x) y_0^2\,dx \geqslant L \cdot \varepsilon^{2\theta + 1 - \frac{\alpha+1}{\gamma}},$$

where L is a constant. Since $\alpha > 2\gamma - 1$, there is a number $\theta > \frac{1}{2}$ such that $2\theta + 1 < \frac{\alpha+1}{\gamma}$.

Thus,

$$\lambda_1(Q_\varepsilon) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q_\varepsilon, y] \leqslant R[Q_\varepsilon, y_0],$$

$$\inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \leqslant \lim_{\varepsilon \to 0} \lambda_1(Q_\varepsilon) \leqslant \lim_{\varepsilon \to 0} R[Q_\varepsilon, y_0] = -\infty.$$

References

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