The Asymptotic of Unboudedly Continuable to the Right Solutions of the Ordinary Differential Equation of Second Order

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We consider the second order ordinary differential equation of the form:

$$F(t,y,y',y'')=\sum_{k=1}^{n}p_k(t)y^{\alpha_k}|y'|^{\beta_k}|y''|^{\gamma_k}=0, \tag{1}$$

$n\in\mathbb{N}$, $n\geq 2$, $\alpha_k,\beta_k,\gamma_k\in\mathbb{R}$, $\sum\limits_{k=1}^{n}|\gamma_k|\neq 0$, $p_k\in\mathrm{C}([a;+\infty),a>0;\mathbb{R})$ $(k=\overline{1,n})$, $p_i(t)\neq 0$ $(i=\overline{1,s}$ for some $2\leq s\leq n)$.

We investigate the question of the existence and asymptotic behavior (as $t\to+\infty$) of unboudedly continuable to the right solutions (R-solutions) $y(t)$ of equation (1) and the derivatives $y'(t)$, $y''(t)$ of these solutions.

Earlier in [3] we have considered a similar question of the asymptotic behavior of solutions of equation of the form (1) when $\sum\limits_{k=1}^{n}|\gamma_k|=0$, that is when equation (1) is a first order differential equation.

The main result is obtained under the assumption that there exists a function $v\in\mathrm{C}^2([t_1;+\infty),t_1>a;\mathbb{R})$ which possesses the following properties:

(A) $v(t)>0$, $v''(t)\neq 0$ on $[t_1;+\infty)$,

$$\lim_{t\to+\infty}v(t)=0\vee+\infty;$$

(B)

$$\lim_{t\to+\infty}\frac{v''(t)v(t)}{(v'(t))^2}=\mu\ \ (0\neq\mu\in\mathbb{R});$$

(C)

$$\lim_{t\to+\infty}\frac{p_i(t)v^{\alpha_i}(t)|v'(t)|^{\beta_i}|v''(t)|^{\gamma_i}}{p_1(t)v^{\alpha_1}(t)|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}}=c_i\ \ (0\neq c_i\in\mathbb{R},\ \ i=\overline{1,s}),$$

$$\sum_{i=1}^{s}\gamma_i c_i\neq 0,$$

$$\lim_{t\to+\infty}\frac{p_j(t)v^{\alpha_j}(t)|v'(t)|^{\beta_j}|v''(t)|^{\gamma_j}}{p_1(t)v^{\alpha_1}(t)|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}}=0\ \ (j=\overline{s+1,n}).$$

The following lemma is valid.

Lemma. *Let in the relation*

$$\Phi(t,x_1,x_2,x_3)=0, \tag{2}$$

$(t,x_1,x_2,x_3)\in H$, $H=[a;+\infty)\times\prod\limits_{k=1}^{3}H_k$, $H_k=[-h_k;h_k]$, $a\in\mathbb{R}$, $h_k>0$ $(k=1,2,3)$, *the function* $\Phi:H\to\mathbb{R}$ *satisfy the conditions:*

1) $\Phi,\frac{\partial\Phi}{\partial x_1},\frac{\partial\Phi}{\partial x_2},\frac{\partial^2\Phi}{\partial x_3^2}\in \mathrm{C}(H;\mathbb{R})$;

2)
$$\lim_{t\to+\infty}\sup_{(x_1;x_2)\in H_1\times H_2}|\Phi(t,x_1,x_2,0)|=0;$$

3)
$$\lim_{t\to+\infty}\frac{\partial\Phi}{\partial x_3}(t,0,0,0)=A_1\neq 0;$$

4)
$$\sup_{D}\left|\frac{\partial^2\Phi}{\partial x_3^2}(t,x_1,x_2,x_3)\right|=A_2<+\infty.$$

Then in some domain $H^*=H_0\times H_3^*$, $H_0=[t_0;+\infty)\times\prod\limits_{k=1}^{2}H_k^*$, $H_k^*=[-h_k^*;h_k^*]$ $(k=1,2,3)$, *where* t_0 *and* h_k^* *satisfy the inequality* $t_0\geq a$, $0<h_k^*\leq h_k$, $\frac{4A_2h_3^*}{|A_1|}<1$, *relation* (2) *defines a unique function* $x_3:H_0\to\mathbb{R}$ *that satisfies the conditions:*

$$x_3,\frac{\partial x_3}{\partial x_1},\frac{\partial x_3}{\partial x_2}\in \mathrm{C}(H_0;\mathbb{R}),\quad \Phi(t,x_1,x_2,x_3(t,x_1,x_2))\equiv 0,\quad \lim_{t\to+\infty}x_3(t,0,0)=0$$

and

$$x_3(t,x_1,x_2)\sim-\frac{\Phi(t,x_1,x_2,0)}{\frac{\partial\Phi}{\partial x_3}(t,x_1,x_2,0)}.$$

The following theorem was obtained using the above lemma and the results from [1,2,4].

Theorem. *Let there exist a function* $v\in \mathrm{C}^2([t_1;+\infty),t_1>a;\mathbb{R})$ *which possesses the properties* (A)–(C). *Then for the R-solution* $y(t)$ *of the differential equation* (1) *with the asymptotic representation*

$$y^{(k)}(t)\sim v^{(k)}(t)\quad (k=\overline{0,2}) \tag{3}$$

to exist it is necessary, and if the roots λ_1, λ_2 *of the algebraic equation*

$$\lambda^2+\left(1+\frac{m\sum\limits_{i=1}^{s}(\beta_i+\gamma_i)c_i}{\sum\limits_{i=1}^{s}\gamma_ic_i}\right)\lambda+\frac{m\sum\limits_{i=1}^{s}(\alpha_i+\beta_i+\gamma_i)c_i}{\sum\limits_{i=1}^{s}\gamma_ic_i}=0$$

have the property $\operatorname{Re}\lambda_k\neq 0$ $(k=1,2)$, *then it is also sufficient that* $\sum\limits_{i=1}^{s}c_i=0$.

Moreover, if $\operatorname{sign}(\operatorname{Re}\lambda_1)\neq\operatorname{sign}(\operatorname{Re}\lambda_2)$, *then there exists a one-parametric set of R-solutions with the asymptotic representation* (3); *if in some suburb of* $+\infty$

$$\operatorname{sign}(\operatorname{Re}\lambda_1)=\operatorname{sign}(\operatorname{Re}\lambda_2)\neq\operatorname{sign}(v'(t)),$$

then there exists a two-parametric set of R-solutions with the asymptotic representation (3).

References

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