Control Problem of Asynchronous Spectrum of a Linear Periodic System with a Degenerate Block of Mean Value of Coefficient Matrix

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We consider the linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where $A(t)$ is a continuous periodic $n \times n$-matrix with a modulus of frequencies Mod, B is the constant $r \times n$-matrix, u is the input. Various problems of control theory of linear systems have been studied in many works (see, for example, [6]). In this works it is assumed, as a rule, that the set of frequencies of the solution and the system itself coincide.

At the same time, as shown by X. Masser [5], Ya. Kurzveil and O. Veivoda [4], etc., the system of ordinary differential periodic (almost periodic) equations can have solutions, the intersection of the frequency module of which with the frequency module of the system is trivial. Later such solutions were named strongly irregular, and their frequency spectrum is asynchronous, and describable oscillations are asynchronous. Note that in the case of a periodic system the irregularity means the incommensurability of the periods of the solution and the system.

In what follows, as a control of $u(\,\cdot\,)$ in system (1) we will use continuous on real axis of periodic r-vector-functions, set of exponentials of which $\operatorname{Exp}(u)$ is contained in the frequency modulus $\operatorname{Mod}(A)$ coefficient matrices.

Then, as applied to system (1), the control problem of asynchronous spectrum with a target set L is as follows: select this program control

$$u = U(t)$$

from the indicated admissible set, so that the system

$$\dot{x} = A(t)x + Bu(t)$$

has a strongly irregular periodic solution with a given spectrum frequency L (target set).

The solvability of the formulated problem for system (1) with program control and zero mean value of the matrix coefficients were studied in the work [3]. In this report, we give a solution of the problem of control of the asynchronous spectrum for system (1), the average value of the matrix of coefficients of which has a degenerate non-zero left upper diagonal block, and the rest of the blocks are zero.

Let $P = (p_{ij})$, $i = \overline{1,n}$, $j = \overline{1,m}$, – some matrix and $1 \leq k_1 < \cdots < k_s \leq n$, $1 \leq l_1 < \cdots < l_q \leq m$ – two ordered sequences of natural numbers. Let $P^{l_1 \cdots l_q}_{k_1 \cdots k_s}$ be a block of the matrix P, standing at the intersection of rows with numbers $k_1, \ldots, k_s$ and columns with numbers $l_1, \ldots, l_q$.

Let $P = (p_{ij})$, $i = \overline{1,n}$, $j = \overline{1,m}$, be some square matrix and $1 \leq k_1 < \cdots < k_s \leq n$, $1 \leq l_1 < \cdots < l_q \leq m$ be two ordered sequences of natural numbers. By $P^{l_1 \cdots l_q}_{k_1 \cdots k_s}$ we denote the $s \times q$-matrix, standing at the intersection of rows with numbers $k_1, \dots, k_s$ and columns with numbers $l_1, \dots, l_q$ of matrix P.

For continuous on $\mathbb{R}$ ω-periodic real-valued matrix $F(t)$, we determine the mean value $\widehat{F} = \frac{1}{\omega}\int\limits_0^\omega F(t)dt$ and the oscillating part $\widetilde{F}(t) = F(t) - \widehat{F}$. Let $\operatorname{Mod}(F)$ be a frequency modulus of the matrix $F(t)$, i.e. the set of all possible linear combinations with integer coefficients of Fourier exponents of this matrix. By $\operatorname{rank}_{\rm col} F$ we denote the column rank of the matrix $F(t)$, i.e. largest number of linearly independent columns. Similarly, it is also possible to determine the row rank of a matrix. Let us note that in the general case, the row and column ranks of the matrix $F(t)$ are not required match. We will talk that $F(t)$ is a matrix of incomplete column rank, if the column rank is less than number of columns.

Further, we assume that the rank of the constant rectangular matrix B under control is not the maximum and row with numbers $k_1, \dots, k_d$, $1 \le k_1 < \dots < k_d \le n$ zero, i.e.,

$$\operatorname{rank} B = r_1 < r, \ \ B^{1\cdots r}_{k_1\cdots k_d} = 0 \ \ (d = n - r_1). \tag{2}$$

The last restriction is not a loss of generality of reasoning, so we can achieve this with the help of a linear system transformations (1) using elementary algorithms matrix row transformations.

We also assume that the mean value of the coefficient matrix is a result of permuting rows and columns, we can represent in the form

$$\begin{pmatrix} \widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d} & \widehat{A}^{k_{d+1}\cdots k_n}_{k_1\cdots k_d} \\ \widehat{A}^{k_1\cdots k_d}_{k_{d+1}\cdots k_n} & \widehat{A}^{k_{d+1}\cdots k_n}_{k_{d+1}\cdots k_n} \end{pmatrix} = \begin{pmatrix} \widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d} & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d} = \operatorname{diag}(\widehat{a}_{k_1\,k_1}, \dots, \widehat{a}_{k_d\,k_d}), \tag{3}$$

and $\widehat{a}_{k_1\,k_1} \cdots \widehat{a}_{k_d\,k_d} = 0$. The last condition means that among the diagonal elements of the block $\widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d}$ are null. It is possible to assume that they are at the beginning of the diagonal

$$\widehat{a}_{k_{1+i-1}\,k_1+i-1} = 0, \ \ i = \overline{1,m}, \ \ 1 \le m < d, \tag{4}$$

and the rest of the elements are non-zero. In the opposite case this can be achieved with the help of linear non-degenerative transformation system (1), which is equivalent to permuting the first d equations in the required order.

Let $k_{d+1}, \dots, k_n$, $1 \le k_{d+1} < \dots < k_n \le n$ be the numbers of non-zero rows of a matrix B. Then, taking into account the numbering of zero and non-zero rows of this matrix to simplify the recording, we take the following notations:

$$A_{11}(t) = A^{k_1\cdots k_d}_{k_1\cdots k_d}(t), \quad A_{12}(t) = A^{k_{d+1}\cdots k_n}_{k_1\cdots k_d}(t).$$

Through $\widetilde{A}^{(1)}_{11}(t)$ we denote $d \times m$-matrix composed of the first m columns of the $d \times d$-block $\widetilde{A}_{11}(t)$. Let's construct $d \times (m + r_1)$-matrix $\widetilde{A}_*(t) = \Big[\widetilde{A}^{(1)}_{11}(t) \quad A_{12}(t)\Big]$.

We have the following

Theorem. *For the linear systems* (1), (2)–(4), *the problem of control of the asynchronous spectrum with target set L is solvable if and only if $L = \{0\}$ and the inequality*

$$\operatorname{rank}_{\rm col} \widetilde{A}_*(t) < r_1 + m$$

is true.

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