The Asymptotic Representation of $P_\omega(Y_0, Y_1, 1)$-Solutions of Second Order Differential Equations with the Product of Regularly and Rapidly Varying Functions in its Right-Hand Side

O. O. Chepok

K. D. Ushinsky South Ukrainian National Pedagogical University, Odessa, Ukraine
E-mail: olachepok@ukr.net

We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y')\varphi_1(y). \tag{1}$$

In this equation the constant α_0 is responsible for the sign of the equation, functions $p : [a,\omega[\to]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ $(i \in \{0,1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is the some one-sided neighborhood of Y_i.

We also suppose that function φ_1 is a regularly varying as $y \to Y_1$ function of the index σ_1 [7, pp. 10–15], function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y') \neq 0 \ \text{ as } \ y' \in \Delta_{Y_0}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \varphi_0(y') \in \{0, +\infty\}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \frac{\varphi_0(y')\varphi_0''(y')}{(\varphi_0'(y'))^2} = 1. \tag{2}$$

It follows from conditions (2) that the following statements are true

$$\frac{\varphi_0'(y')}{\varphi_0(y')} \sim \frac{\varphi_0''(y')}{\varphi_0'(y')} \ \text{ as } \ y' \in \Delta_{Y_0}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \frac{y'\varphi_0'(y')}{\varphi_0(y')} = \pm\infty. \tag{3}$$

Also it follows from the above conditions (3) that the function φ_0 and its first-order derivative are rapidly varying functions as the argument tends to Y_0 [1].

So (1) is the second order differential equation that contains in the right-hand side the product of a regularly varying function of unknown function and a rapidly varying function of the first derivative of the unknown function.

In the previous works (see, for example [2]) we obtained results for the second order differential equation containing a rapidly varying function of unknown function and a regularly varying function of its first derivative.

For equation (1) we consider the following class of solutions.

Definition 1. The solution y of the equation (1), that is defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$, if the following conditions take place

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

In the work we establish the necessary and sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) in case $\lambda_0=1$ and find asymptotic representations of such solutions and its first order derivatives as $t\uparrow\omega$.

According to the properties of such $P_\omega(Y_0,Y_1,1)$-solutions (see, for example, [4]) we have that

$$\lim_{t\uparrow\omega}\frac{y'(t)}{y(t)}=\lim_{t\uparrow\omega}\frac{y''(t)}{y'(t)}\,,$$

and

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)}=\pm\infty,\quad \pi_\omega(t)=\begin{cases}t & \text{as } \omega=+\infty,\\ t-\omega & \text{as } \omega<+\infty,\end{cases}$$

So we have that each such $P_\omega(Y_0,Y_1,1)$-solution and its first-order derivative are rapidly varying functions as $t\uparrow\omega$ and this case of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions is the most difficult.

Let the solution y of equation (1) is a $P_\omega(Y_0,Y_1,1)$-solution. Note that the function $y(t(y'))$, where $t(y')$ is an inverse function to $y'(t)$, is a regularly varying function of the index 1 as $y'\to Y_0$ $(y'\in\Delta_{Y_0})$.

Indeed, the following statement is true

$$\lim_{y'\to Y_1}\frac{y'(y(t(y')))'}{y(t(y'))}=\lim_{y'\to Y_1}\frac{(y'(t(y')))^2}{y(t(y'))y''(t(y'))}=1.$$

Definition 2. Let $Y\in\{0,\infty\}$, Δ_Y be some one-sided neighborhood of Y. A continuous-differentiable function $L:\Delta_Y\to]0;+\infty[$ is called [6, pp. 2-3] a normalized slowly varying function as $z\to Y$ $(z\in\Delta_Y)$ if the next statement is true

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}}\frac{yL'(y)}{L(y)}=0.$$

Definition 3. We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition S as $z\to Y$, if for any continuous differentiable normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to]0;+\infty[$ the next relation is valid

$$\theta(zL(z))=\theta(z)(1+o(1))\ \text{ as }\ z\to Y\ \ (z\in\Delta_Y).$$

Definition 4. Let's define that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L_0:\Delta_Y\to]0;+\infty[$ satisfies the condition S_1 as $z\to Y$ if for any finite segment $[a;b]\subset]0;+\infty[$ the next inequality is true

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}}\Big|\ln|z|\cdot\Big(\frac{L(\lambda z)}{L(z)}-1\Big)\Big|<+\infty\ \text{ for all }\ \lambda\in[a;b].$$

Note that

$$\Phi_0(y')=\operatorname{sign}y_1^0\int_{B_0}^{y}|s|^{\frac{1}{\sigma_1-2}}\varphi_0^{\frac{1}{\sigma_1-2}}(s)\,ds,\quad B_0=\begin{cases}y_1^0, & \text{if } \displaystyle\int_{y_0^1}^{Y_0}|s|^{\frac{1}{\sigma_1-2}}\varphi_0^{\frac{1}{\sigma_1-2}}(s)\,ds=\pm\infty,\\ Y_0, & \text{if } \displaystyle\int_{y_1^0}^{Y_0}|s|^{\frac{1}{\sigma_1-2}}\varphi_0^{\frac{1}{\sigma_1-2}}(s)\,ds=const,\end{cases}$$

$$\theta_1(z)=\varphi_1(z)|z|^{-\sigma_1},\quad Z_0=\lim_{\substack{y'\to Y_0\\ y'\in\Delta_{Y_0}}}\Phi_0(y'),\quad \Phi_1(y')=\int_{B_0}^{y'}\Phi_0(s)\,ds,\quad Z_1=\lim_{\substack{y'\to Y_0\\ y\in\Delta_{Y_0}}}\Phi_1(y),$$

$$I_0(t)=\int\limits_{A_0}^{t} p^{\frac{1}{2-\sigma_1}}(\tau)\,d\tau,\quad A_0=\begin{cases} a, & \text{if } \int\limits_a^{\omega} p^{\frac{1}{2-\sigma_1}}(\tau)\,d\tau=+\infty,\\ \omega, & \text{if } \int\limits_a^{\omega} p^{\frac{1}{2-\sigma_1}}(\tau)\,d\tau<+\infty\end{cases}$$

in the case $\lim\limits_{t\uparrow\omega} I_0(t)=Z_0$ and $\operatorname{sign} I_0(t)=\operatorname{sign}\Phi_0(y)$, let

$$I_1(t)=\int\limits_{A_1}^{t}\frac{1}{\Phi_0^{-1}(I_0(\tau))))}\,d\tau,\quad A_1=\begin{cases} b, & \text{if } \int\limits_b^{\omega}\frac{1}{\Phi_0^{-1}(I(\tau))))}\,d\tau=\pm\infty,\\ \omega, & \text{if } \int\limits_b^{\omega}\frac{1}{\Phi_0^{-1}(I_0(\tau))))}\,d\tau=const, \quad b\in[a;\omega[\,,\end{cases}$$

$$I_2(t)=-\int\limits_{A_2}^{t}\Big(\frac{I_0(\tau)}{I_1(\tau)}\Big)\,d\tau,\quad A_2=\begin{cases} b, & \text{if } \int\limits_b^{\omega}\Big(\frac{I_0(\tau)}{I_1(\tau)}\Big)\,d\tau=\pm\infty,\\ \omega, & \text{if } \int\limits_b^{\omega}\Big(\frac{I_0(\tau)}{I_1(\tau)}\Big)\,d\tau=const.\end{cases}$$

Note 1. The following statements are true:

1)
$$\Phi_0(z)=(\sigma_1-1)\,\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(z)}{\varphi_0'(z)}\,[1+o(1)] \ \text{ as } \ z\to Y_0 \ \ (z\in\Delta_{Y_0}).$$

From this we have
$$\operatorname{sign}(\varphi_0'(z)\Phi_0(z))=\operatorname{sign}(\sigma_1-1) \ \text{ as } \ z\in\Delta_{Y_0}.$$

2)
$$\Phi_1(z)=\frac{\Phi_0^2(z)}{z\Phi_0'(z)}\,[1+o(1)] \ \text{ as } \ z\to Y_1 \ \ (z\in\Delta_{Y_0}).$$

From this we have
$$\operatorname{sign}(\Phi_1(z))=y_0^1 \ \text{ as } \ z\in\Delta_{Y_0}.$$

3) The functions Φ_0^{-1} and Φ_1^{-1} exist and are slowly varying functions as inverse to rapidly varying functions as the arguments tend to Y_0 functions.

4) The function Φ_1' (Φ_1^{-1}) is a regularly varying function of the index 1 as the argument tends to Y_0.

Note 2. The function $\theta_1(y(t(y')))$ is a slowly varying function for $y'\to Y_0$ $(y'\in\Delta_{Y_0})$ as a composition of regularly and slowly varying functions as $y'\to Y_0$ $(y'\in\Delta_{Y_0})$.

Let's consider the function $\theta_1(y(I_1^{-1}(z)))$, where $I_1^{-1}(z)$ is the function inverse to the function $I_1(t)$, and it can be proved that $\theta_1(y(I_1^{-1}(z)))$ is a slowly varying function as $z\to Z_1$.

Indeed,

$$\lim_{z\to Z_1}\frac{z(\theta_1(y(I_1^{-1}(z))))'}{\theta_1(y(I_1^{-1}(z)))}=\lim_{z\to Z_0}\left(\frac{z\theta_1'(y(I_1^{-1}(z)))}{\theta_1(y(I_1^{-1}(z)))}\cdot\frac{y'(I_1^{-1}(z))}{I_1'(I_0^{-1}(z))}\right)$$
$$=\lim_{z\to Z_0}\left(\frac{y(I_1^{-1}(z))\theta_1'(y(I_1^{-1}(z)))}{\theta_1(y(I_1^{-1}(z)))}\cdot\frac{y(I_0^{-1}(z))\cdot y'(y'^{-1}(y'(I_1^{-1}(z))))}{(y(y'^{-1}(y'(I_1^{-1}(z)))))^2}\right.$$
$$\left.\times\frac{\tilde{\Phi}(y'(I_1^{-1}(z)))}{y'(I_1^{-1}(z))\widetilde{\Phi}'(y'(I_1^{-1}(z)))}\cdot\frac{z\widetilde{\Phi}'(y'(I_1^{-1}(z)))}{I_1'(I_1^{-1}(z))\tilde{\Phi}(y'(I_1^{-1}(z)))}\right)=0.$$

Let the function Φ_1^{-1} satisfy the condition S, and we have that

$$y'(t)=\Phi_1^{-1}(I_1(t))[1+o(1)]\ \text{ as }\ t\uparrow\omega.$$

The following theorem takes place.

Theorem 1. *Let $\sigma_1\in R\setminus\{1\}$, the function θ_1 satisfy the condition S, and the functions θ_1 and $\Phi_1^{-1}\cdot\frac{\Phi_1'}{\Phi_1}(\Phi_1^{-1})$ satisfy the condition S_1. Then for the existence of $P_\omega(Y_0,Y_1,1)$-solutions of equation* (1) *it is necessary, and if the following condition takes place*

$$(\sigma_1-2)\cdot y_0^0 I_0(t)\cdot I_2(t)>0\ \ as\ \ t\in[a;\omega[\,, \tag{4}$$

and there is a finite or infinite limit

$$\frac{\sqrt{\left|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\right|}}{\ln|I_1(t)|},$$

then it is sufficient the fulfillment of the next conditions

$$y_0^0\alpha_0>0,\quad \lim_{t\uparrow\omega}\Phi_1^{-1}(I_2(t))=Y_0,\quad \lim_{t\uparrow\omega}I_2(t)=Z_1, \tag{5}$$

$$\lim_{t\uparrow\omega}\frac{\Phi_1'(\Phi_1^{-1}(I_2(t)))}{I_1(t)I_2'(t)}=-1, \tag{6}$$

$$y_0^0\cdot I_1(t)<0\ \ as\ \ t\in]b;\omega[\,,\quad \lim_{t\uparrow\omega}\frac{-1}{I_1(t)}=Y_1, \tag{7}$$

$$\lim_{t\uparrow\omega}\frac{I_2(t)\cdot I_0'(t)\cdot\theta_1^{\frac{1}{2-\sigma_1}}(-\frac{1}{I_1(t)})}{\Phi_1'(\Phi_1^{-1}(I_2(t)))I_2'(t)}=1. \tag{8}$$

Moreover, for each such solution the next asymptotic representations as $t\uparrow\omega$ take place:

$$y'(t)=\Phi_1^{-1}(I_1(t))[1+o(1)],\quad y(t)=\frac{I_2'(t)I_1(t)}{I_2(t)\Phi_1'(\Phi_1^{-1}(I_1(t)))}[1+o(1)]. \tag{9}$$

During the proof of Theorem 1, equation (1) is reduced by a special transformation to the equivalent system of quasilinear differential equations. The limit matrix of coefficients of this system has real eigenvalues of different signs.

We obtain that for this system of differential equations all the conditions of Theorem 2.2 in [5] take place. According to this theorem, the system has a one-parameter family of solutions $\{z_i\}_{i=1}^2:[x_1,+\infty[\to\mathbb{R}^2$ $(x_1\geq x_0)$, that tends to zero as $x\to+\infty$.

Any solution of the family gives raise to such a solution y of equation (1) that, together with its first derivative, admit the asymptotic images (9) as $t\uparrow\omega$. From these images and conditions (5)–(8) it follows that these solutions are $P_\omega(Y_0,Y_1,1)$-solutions.

References

[1] N. H. Bingham, C. M.Goldie and J. L. Teugels, *Regular Variation.* Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge, 1987.

[2] O. O. Chepok, Asymptotic representations of a class of regularly varying solutions of differential equations of the second order with rapidly and regularly varying nonlinearities. *Mem. Differ. Equ. Math. Phys.* **74** (2018), 79–92.

[3] O. O. Chepok, Asymptotic representations of regularly varying $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the second-order differential equation, which contains the product of different types of nonlinearities of an unknown function and its derivative. *Nonlinear Oscil.* **25** (2022), no. 1, 133–144.

[4] V. M. Evtukhov, Asymptotic properties of solutions of n-order differential equations. (Russian) *Ukrainian Mathematics Congress 2001 (Ukrainian)*, 15–33, *Natsīonal. Akad. Nauk Ukraïni, Īnst. Mat., Kiev*, 2002.

[5] V. M. Evtukhov and A. M. Samoilenko, Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point. *Ukrainian Math. J.* **62** (2010), no. 1, 56–86.

[6] V. Marić, *Regular Variation and Differential Equations.* Lecture Notes in Mathematics, 1726. Springer-Verlag, Berlin, 2000.

[7] E. Seneta, *Regularly Varying Functions.* Lecture Notes in Mathematics, Vol. 508. Springer-Verlag, Berlin–New York, 1976.