On the Solvability of the Cauchy Problem for Second Order Functional Differential Equations with an Alternating Coefficient

Eugene Bravyi

Perm National Research Polytechnic University, Perm, Russia

E-mail: bravyi@perm.ru

There are many papers devoted to the solvability of the Cauchy problem in the non-Volterra case [1–15]. If the functional operators in the equation don't satisfy the delay conditions, the solvability of the Cauchy problem requires some smallness of these functional operators.

We consider the Cauchy problem for functional differential equations with an alternating coefficient

$$\begin{cases} \ddot{x}(t) = a(t-t_0)x(h(t)) + f(t), \quad t \in [0,1], \\ x(0) = c_0, \quad \dot{x}(0) = c_1, \end{cases} \tag{1}$$

where $a \in \mathbb{R}$, $t_0 \in [0,1]$, $h : [0,1] \to [0,1]$ is a measurable function, $f \in \mathbf{L}[0,1]$, c_0, $c_1 \in \mathbb{R}$. We say that a function $x : [0,1] \to \mathbb{R}$ is a solution of problem (1) if x and the derivative $\dot{x}$ are absolutely continuous on the interval $[0,1]$ and x satisfies the functional differential equation of the problem almost everywhere on $[0,1]$ and satisfies the initial conditions $x(0) = c_0$ and $\dot{x}(0) = c_1$.

Using ideas of [8,9], we obtain necessary and sufficient conditions for the Cauchy problem

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [0,1], \\ x(0) = c_0, \quad \dot{x}(0) = c_1 \end{cases} \tag{2}$$

to be uniquely solvable for all linear positive operators T^+, $T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that

$$\begin{aligned} (T^+\mathbf{1}\,)(t) &= \begin{cases} a(t-t_0) & \text{if } a(t-t_0) \geq 0, \\ 0, & \text{otherwise,} \end{cases} \\ (T^-\mathbf{1}\,)(t) &= \begin{cases} -a(t-t_0) & \text{if } a(t-t_0) < 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

Here $\mathbf{1}\, : [0,1] \to \mathbb{R}$, $\mathbf{1}\,(t) = 1$, is the unit function, $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ are the spaces of all continuous and integrable functions with the standard norms respectively, an operator is called *positive* if it maps each non-negative function into almost everywhere non-negative one.

We also need the following notation.

Let $t_{0*} \approx 0,47$ be a solution of the equation

$$\frac{6}{t_0^2(3-t_0)} = \frac{6}{2-3t_0}\,,$$

$t_0^* \approx 0,54$ be a solution of the equation

$$\frac{24}{(3t_0-1)^2} = \frac{6}{(1-t_0)^3}\,.$$

Denote

$$
\begin{aligned}
q_1 = q_1(t_0,t_1,t_3) &= (t_0-t_1)^3 - 3(1-t_1)(t_0-t_3)^2 + 3t_0 - 1,\\
q_2 = q_2(t_0,t_1,t_3) &= t_1^2(3-t_0-2t_3)(t_0-t_3)^2(3t_0-t_1) - (3t_0-1)(t_1-t_3)^2(3t_0-t_1-2t_3),\\
r_1 = r_1(t_0,t_1,t_3) &= \frac{t_1^2(3t_0-t_1)+3(t_0-t_3)^2(t_1-1)}{6}\,,\\
r_2 = r_2(t_0,t_1,t_3) &= \frac{(t_1(t_0+2t_3)(3t_0-t_1)+(3t_1-t_0-2t_3)(1+t_1-3t_0))(t_0-t_3)^2(t_1-1)}{36}\,,
\end{aligned}
$$

$$
A^+(t_0) = \begin{cases} \dfrac{6}{(1-t_0)^3} & \text{if } t_0\in[0,t_0^*],\\[2mm] \min\limits_{0<t_3\le t_1<t_0} \dfrac{3(q_1+\sqrt{q_1^2+4\,q_2})}{q_2} & \text{if } t_0\in(t_0^*,1], \end{cases}
$$

$$
A^-(t_0) = \begin{cases} \min\limits_{0<t_3\le t_1<t_0} \left\{ \dfrac{3(r_1-\sqrt{r_1^2-4\,r_2})}{r_2}\,,\ \dfrac{6}{t_0^2(3-t_0)} \right\} & \text{if } t_0\in[0,t_{0*}),\\[2mm] \dfrac{6}{t_0^2(3-t_0)} & \text{if } t_0\in[t_{0*},1]. \end{cases}
$$

Theorem. *Problem* (2) *is uniquely solvable for all linear positive operator* T^+, $T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ *satisfied conditions* (3) *if and only if*

$$
-A^-(t_0) < a < A^+(t_0). \tag{4}
$$

Corollary. *Problem* (1) *is uniquely solvable for every measurable function* $h:[0,1]\to[0,1]$ *if and only if condition* (4) *holds.*

Example. For $t_0\in[1/5,t_0^*]$, we have

$$
A^-(t_0) = \frac{6}{t_0^2(3-t_0)}\,,\quad A^+(t_0) = \frac{6}{(1-t_0)^3}\,.
$$

In particular, the problem

$$
\begin{cases} \ddot{x}(t) = a\Big(t-\dfrac{1}{2}\Big)x(h(t)) + f(t), \quad t\in[0,1],\\ x(0)=c_0, \quad \dot{x}(0)=c_1 \end{cases}
$$

is uniquely solvable for every measurable function $h:[0,1]\to[0,1]$ if and only if

$$
-\frac{48}{5} < a < 48.
$$

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