Fredholm Boundary-Value Problem for the System of Fractional Differential Equations with Caputo Derivative

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In the space $C[a,b]$, $-\infty < a < b < +\infty$, we consider a linear boundary-value problem for the system of fractional differential equations

$$ {}^C\mathrm{D}^{\alpha}_{a+}x(t) = A(t)x(t) + f(t), \tag{1}$$
$$ lx(\,\cdot\,) = q, \tag{2}$$

where ${}^C\mathrm{D}^{\alpha}_{a+}$ is the left Caputo fractional derivative of order α $(0 < \alpha < 1)$ [6,7,14]

$$ {}^C\mathrm{D}^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(m-\alpha)} \int\limits_a^t \frac{x^{(m)}(s)}{(t-s)^{\alpha-m+1}}\,\mathrm{d}s,$$

$A(t)$ is an $(n\times n)$-matrix and $f(t)$ is an n-vector, whose components are real functions continuous on $[a,b]$, $l = \mathrm{col}(l_1, l_2, \dots, l_p) : C[a,b] \to \mathbb{R}^p$ is bounded linear vector functional, $l_\nu : C[a,b] \to \mathbb{R}$, $\nu = \overline{1,p}$, $q = \mathrm{col}(q_1, q_2, \dots, q_p) \in \mathbb{R}^p$.

Using the results [1,2,5,15], obtained as a generalization of the classical methods of the theory of periodic boundary-value problems in the theory of oscillations (see [10–13]), we consider the questions of finding necessary and sufficient conditions of solvability and determine a general form of solutions of the boundary-value problem for the systems of fractional differential equations (1), (2). Let us first consider the general solution of system (1) of the form

$$ x(t) = X(t)c + \overline{x}(t) \;\; \forall\, c \in \mathbb{R}^n, \tag{3}$$

where $X(t)$ is the fundamental solution $(n\times n)$-matrix of the homogeneous system (1) $(f = 0)$, whose column vectors constitute a fundamental system of solutions to the homogeneous system (1) and $\overline{x}(t)$ is an arbitrary special solution of the inhomogeneous system (1). The required special solution $\overline{x}(t)$ can be chosen as a solution of the system of linear Volterra integral equation of the second kind

$$ \overline{x}(t) = g(t) + \int\limits_a^t K(t,s)\overline{x}(s)\,ds, \tag{4}$$

$$ g(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{f(s)}{(t-s)^{\gamma}}\,ds, \quad K(t,s) = \frac{A(s)}{\Gamma(\alpha)(t-s)^{\gamma}}\,, \tag{5}$$

$0 < \gamma = 1 - \alpha < 1$.

The solution of the system of equations (4) can be found by different methods. We apply the approach described in [3,4]. In the Hilbert space $L_2[a,b]$, we show that system (4) with unbounded kernel $K(t,s)$ (5) can be reduced to an equivalent system with square summable kernel. To do this, we consider iterated kernels $K_m(t,s)$, $m\in\mathbb{N}$, given by the recurrence relations

$$K_{m+1}(t,s)=\int\limits_s^t K(t,\xi)K_m(\xi,s)\,\mathrm{d}\xi,\ \ K_1(t,s)=K(t,s).$$

The iterated kernels $K_m(t,s)$ have the same structure as weakly singular kernel $K(t,s)$ (5) but the number γ is replaced with the number $1-m(1-\gamma)$ which is negative for sufficiently large m. Therefore (see [9, p. 34]), for all m by which the condition

$$m>\frac{1}{2(1-\gamma)} \tag{6}$$

is satisfied, the kernels $K_m(t,s)$ are square summable.

System (4) can be reduced to a similar system with the kernel $K_m(t,s)$

$$\overline{x}(t)=g^m(t)+\int\limits_a^t K_m(t,s)\overline{x}(s)\,ds, \tag{7}$$

$$g^m(t)=g(t)+\sum_{l=1}^{m-1}\int\limits_a^t K_l(t,s)g(s)\,ds.$$

We apply the approach described in [3, 4] to the study of system (7) and show that it can be reduced to the system

$$\Lambda z=g, \tag{8}$$

where the vectors z, g and the block matrix Λ have the form

$$z=\operatorname{col}(x_1,x_2,\dots,x_i,\dots),\quad g=\operatorname{col}(g_1,g_2,\dots,g_i,\dots),$$

$$\Lambda=\begin{pmatrix}\Lambda_{11}&\Lambda_{12}&\cdots&\Lambda_{1i}&\cdots\\ \Lambda_{21}&\Lambda_{22}&\cdots&\Lambda_{2i}&\cdots\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ \Lambda_{i1}&\Lambda_{i2}&\cdots&\Lambda_{ii}&\cdots\\ \vdots&\vdots&\dots&\vdots&\ddots\end{pmatrix},\quad \Lambda_{ij}=\begin{cases}I_n-A_{ij}, & \text{if } i=j;\\ -A_{ij}, & \text{if } i\neq j,\end{cases}$$

$$x_i=\int\limits_a^b x(t)\varphi_i(t)\,dt,\quad g_i=\int\limits_a^b g^m(t)\varphi_i(t)\,dt, \tag{9}$$

$$A_{ij}=\int\limits_a^b\int\limits_a^t K_m(t,s)\varphi_i(t)\varphi_j(s)\,dt\,ds, \tag{10}$$

$\{\varphi_i(t)\}_{i=1}^{\infty}$ is a complete orthonormal system of functions in $L_2[a,b]$.

Here, I_n is the identity matrix of dimensions n, the operator $\Lambda:\ell_2\to\ell_2$ appearing on the left-hand side of the operator equation (8) has the form $\Lambda=I-A$, where $I:\ell_2\to\ell_2$ is the identity operator and $A:\ell_2\to\ell_2$ is a compact Volterra operator (see [8]). Hence, $P_\Lambda=P_\Lambda^*=O$, $\Lambda^+=\Lambda^{-1}$. According to [5], the homogeneous equation (8) ($g=0$) possesses a unique solution $z=0$ and the inhomogeneous equation (8) possesses a unique solution of the form $z=\Lambda^{-1}g$.

According to the Riesz–Fischer theorem, one can find an element $\overline{x} \in L_2[a,b]$ such that the quantities x_i, $i = \overline{1,\infty}$ are the Fourier coefficients of this element. Thus, the following representation is true:

$$\overline{x}(t) = \sum_{i=1}^{\infty} x_i \varphi_i(t) = \Phi(t) z = \Phi(t) \Lambda^{-1} g, \tag{11}$$

where

$$\Phi(t) = (\varphi_1(t), \varphi_2(t), \dots \varphi_i(t), \dots).$$

The element $\overline{x}(t)$ given by relations (11) is the required solution of system (7).

We now return to the problem on the existence of a solution of the boundary-value problem (1), (2) and determine a structure of this solution. Substituting (3) in condition (2), we obtain the following algebraic system for vector c:

$$Qc = b, \tag{12}$$

where a $(p \times n)$-matrix Q and a p-vector b having the forms

$$Q = (lX)(\,\cdot\,), \quad b = q - (l\,\overline{x})(\,\cdot\,). \tag{13}$$

According to the criterion for solvability of system (12) (see [5, p. 65]), the following assertion is true.

Theorem. *The homogeneous boundary-value problem* (1), (2) ($f(t) = 0$, $q = 0$) *possesses a* d_2*-parameter family of solutions*

$$x(t) = X(t) P_{Q_{d_2}} c_{d_2} \ \ \forall\, c_{d_2} \in \mathbb{R}^{d_2}.$$

The inhomogeneous boundary-value problem (1), (2) *is solvable if and only if* d_1 *linearly independent conditions*

$$P_{Q^*_{d_1}} b = 0, \quad d_1 = p - \operatorname{rank} Q$$

are satisfied and possesses a d_2*-parameter family of solutions* $x \in C[a,b]$ *of the form*

$$x(t) = X(t) P_{Q_{d_2}} c_{d_2} + X(t) Q^+ b + \overline{x}(t) \ \ \forall\, c_{d_2} \in \mathbb{R}^{d_2}.$$

Here, $P_{Q_{d_2}}$ is an $(r \times d_2)$-matrix formed by a complete system of d_2 linearly independent columns of the matrix projector P_Q, where P_Q is the projector onto the kernel of the matrix Q, Q^+ is the pseudoinverse Moore–Penrose $(n \times p)$-matrix for the matrix Q, and $P_{Q^*_{d_1}}$ is a $(d_1 \times p)$-matrix formed by the complete system of d_1 linearly independent rows of the matrix projector P_{Q^*}, where P_{Q^*} is the projector onto the cokernel of the matrix Q.

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