Nonlinear Integro-Differential Fredholm Type Boundary Value Problems Not Solved with Respect to the Derivative with Delay

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The study of the linear differential-algebraic boundary value problems is connected with numerous applications of corresponding mathematical models in the theory of nonlinear oscillations, mechanics, biology, radio engineering, the theory of the motion stability. Thus, the actual problem is the transfer of the results obtained in the articles and monographs of S. Campbell, A. M. Samoilenko and O. A. Boichuk on the nonlinear boundary value problems to the integro-differential boundary value problem of Fredholm type not solved with respect to the derivative, in particular, finding the necessary and sufficient conditions of the existence of the desired solutions of the nonlinear integro-differential boundary value problem not solved with respect to the derivative with delay. We found the conditions of the existence and constructive scheme for finding the solutions of the nonlinear integro-differential boundary value problem not solved with respect to the derivative with delay.

We investigate the problem of finding solutions [3]

$$y(t) \in \mathbb{D}^2[0,T], \ \ y'(t) \in \mathbb{L}^2[0,T]$$

of the linear Noetherian $(n \neq v)$ boundary value problem for a system of linear integro-differential equations of Fredholm type not solved with respect to the derivative with delay $[1,3,10]$

$$A(t)y'(t) = B(t)y(t) + C(t)y(h(t)) + \Phi(t) \int\limits_{\Delta}^{T} F(y(s), y(h(s)), y'(s), s)\, ds + f(t), \tag{1}$$

$$y(t) = \varphi(t) \in \mathbb{C}^1[0,\Delta], \ \ \ell y(\,\cdot\,) = \alpha, \ \ \alpha \in \mathbb{R}^v. \tag{2}$$

We seek a solution of the boundary value problem $(1),(2)$ in a small neighborhood of the solution

$$y_0(t) \in \mathbb{D}^2[0,T], \ \ y_0'(t) \in \mathbb{L}^2[0,T]$$

of the Noetherian generating problem

$$A(t)y_0'(t) = B(t)y_0(t) + C(t)y(h(t)), \ \ \ell y_0(\,\cdot\,) = \alpha \tag{3}$$

in the case when the matrix $A(t)$ has a variable rank in $[\Delta,T]$. Here

$$A(t), B(t) \in \mathbb{L}^2_{m\times n}[0,T] := \mathbb{L}^2[0,T] \otimes \mathbb{R}^{m\times n}, \ \ \Phi(t) \in \mathbb{L}^2_{m\times q}[0,T],$$
$$f(t) \in \mathbb{L}^2[0,T], \ \ h(t) : [\Delta,T] \to [0,\Delta].$$

We assume that the matrix $A(t)$ is, generally speaking, rectangular: $m \neq n$. Nonlinear vector-function $F(y(t), y(h(t)), y'(t), t)$ is twice continuously differentiable with respect to the unknowns $y(t)$ and with respect to the derivative $y'(t)$ in a small neighborhood of the solution

$$y_0(t) \in \mathbb{C}[0,T], \ \ y_0(t) \in \mathbb{D}^2[0;T], \ \ y_0'(t) \in \mathbb{L}^2[\Delta;T], \ \ T := (q+1)\,\Delta, \ \ q \in \mathbb{N}$$

to the generating problem (3);

$$\ell y(\,\cdot\,) : \mathbb{D}^2[0,T] \to \mathbb{R}^p$$

is a linear bounded vector functional defined on a space $\mathbb{D}^2[0,T]$. The problem of finding solutions of the boundary value problem (1), (2) in case $A(t) = I_n$ was solved by A. M. Samoilenko and A. A. Boichuk [11]. Thus, the boundary value problem (1), (2) is a generalization of the problem solved by A. M. Samoilenko and A. A. Boichuk and also is a generalization of the Noetherian boundary value problems for systems of differential-algebraic equations [4, 7, 8].

Solution of the generating problem (3) can be determined as solution of the problem

$$A(t)y_0'(t) = B(t)y_0(t) + g(t), \quad g(t) := C(t)\varphi(h(t)) + f(t). \tag{4}$$

Let the differential-algebraic system (4) with the constant-rank matrix $A(t)$ satisfy the conditions of the theorem from the paper [7, p. 15]. Then, in the case of the p-order degeneration, the differential-algebraic system (4) has a solution which can be written in the form

$$y_0(t, c_{\rho_{p-1}}) = X_p(t)c_{\rho_{p-1}} + K\big[g(s), \nu_p(s)\big](t), \ \ t \in [\Delta;T], \ \ c_{\rho_{p-1}} \in \mathbb{R}^{\rho_{p-1}}.$$

There $K[g(s), \nu_p(s)](t)$ is generalized Green's operator of the Cauchy problem for the differential-algebraic system (4) where $\nu_p(t)$ is an arbitrary continuous vector function. Substituting the general solution of the Cauchy problem for the differential-algebraic system (4), namely,

$$y_0(t) := K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](t), \ \ t \in [\Delta;T]$$

into the boundary condition (1), we arrive at the linear algebraic equation

$$P_{X_p^*}(\Delta)\Big\{\varphi(\Delta) - K\big[g(s), \nu_p(s)\big](\Delta)\Big\} = 0, \tag{5}$$

where $P_{X_p^*}(\Delta)$ is an orthoprojector,

$$\begin{aligned} &K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](t) \\ &\qquad := X_p(t)X_p^+(\Delta)\Big\{\varphi(\Delta) - K\big[g(s), \nu_p(s)\big](\Delta)\Big\} + K\big[g(s), \nu_p(s)\big](t), \ \ t \in [\Delta;T]. \end{aligned}$$

Linear bounded vector functional $\ell y(\,\cdot\,)$ present in the form

$$\ell y(\,\cdot\,) = \ell_0 y(\,\cdot\,) + \ell_1 y(\,\cdot\,) : \mathbb{C}[0;T] \to \mathbb{R}^\upsilon,$$

where

$$\ell_0 y(\,\cdot\,) := \int\limits_0^\Delta dW(t)\,y(t) : \mathbb{C}[0;\Delta] \to \mathbb{R}^\upsilon, \quad \ell_1 y(\,\cdot\,) := \int\limits_\Delta^T dW(t)\,y(t) : \mathbb{C}[\Delta;T] \to \mathbb{R}^\upsilon;$$

$W(t)$ is an $(\upsilon \times n)$ matrix whose entries are functions of bounded variation on $[0,T]$, and the integral used to represent linear functionals is understood in the Riemann–Stieltjes sense [3]. Let the differential-algebraic system (4) with the constant-rank matrix $A(t)$ satisfy the conditions of the theorem from the paper [7, p. 15]. Only if condition

$$\ell_0\,\varphi(\,\cdot\,) + \ell_1 K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](\,\cdot\,) = \alpha$$

is satisfied, the general solution of the differential-algebraic system (4)

$$y_0(t) = G\big[f(s), \varphi(s); \alpha\big](t), \quad t \in [\Delta; T]$$

determines the solution of the nonlinear differential-algebraic boundary-value problem (1), (2), where [7]

$$G[f(s), \varphi(s); \alpha](t) := K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](t), \quad t \in [\Delta; T].$$

We found the conditions of the existence and constructive scheme for finding the solutions of the nonlinear integro-differential boundary value problem (1), (2) not solved with respect to the derivative with delay.

Conditions for the solvability of the linear boundary-value problem for systems of differential-algebraic equations (3) with the variable rank of the leading-coefficient matrix and the corresponding solution construction procedure have been found in the paper [9]. In the case of nonsolvability, the nonsingular integro-differential boundary value problems can be regularized analogously [6,12].

The proposed scheme of studies of the nonlinear integro-differential boundary value problems of Fredholm type not solved with respect to the derivative with delay (1), (2) is a generalization of the Noetherian boundary-value problems for systems of differential-algebraic equations [2,4,5,7–9].

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