On Two Generalizations of Exponentially Dichotomous Systems

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1 Introduction. Basic definitions

For a positive integer n, by $\mathcal{M}_n$ we denote the class of linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geqslant 0, \tag{1.1}$$

whose coefficient matrices $A(\,\cdot\,)\colon [0,+\infty) \to \operatorname{End}\mathbb{R}^n$ are piecewise continuous and bounded on the time half-line $t \geqslant 0$. By $C\mathcal{M}_n$ we denote a subclass of the class $\mathcal{M}_n$, consisting of systems with continuous coefficients on the half-line. We identify system (1.1) with its coefficient matrix and write $A \in \mathcal{M}_n$ or $A \in C\mathcal{M}_n$. The linear space of solutions of system (1.1) is denoted by $\mathcal{X}(A)$.

The following definition is well known.

Definition 1.1. A system in $\mathcal{M}_n$ is said to be *exponentially dichotomous* or called a *system with exponential dichotomy on the half-line* if there exist positive constants c_1, c_2 and ν_1, ν_2 and a decomposition of the space $\mathbb{R}^n$ of initial data (at $t = 0$) into a direct sum of subspaces L_- and L_+ (the case of zero dimension of the subspaces not being excluded) such that the solutions $x(\,\cdot\,)$ of the system satisfy the following two conditions:

(1) if $x(0) \in L_-$, then $\|x(t)\| \leqslant c_1 e^{-\nu_1(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$;

(2) if $x(0) \in L_+$, then $\|x(t)\| \geqslant c_2 e^{\nu_2(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$.

The study of this class of systems was initiated in Perron's paper [13]. It was preceded by fundamental works by Hadamard [10] and Bohl [8], who had developed the same key ideas that later transformed to the concept of exponential dichotomy. The above definition was actually given by Maisel' [11], but it was Massera and Schäffer [12] who stated it explicitly for the first time. Systems with exponential dichotomy, are one of the most comprehensively studied classes of linear differential systems, with, in addition, has important application in related branches of the theory of differential equations (see, e.g., [1]).

The efficiency of the notion of exponential dichotomy is in studying the asymptotics of solutions of nonlinear systems that are exponentially dichotomous in the first approximation and in its applications to dynamical systems has served as a reason for diverse generalizations of this notion within the theory of linear differential systems itself and beyond, e.g. in the theory of evolution operators and in the theory of linear extensions dynamical systems. We do not give any references to papers dealing with such generalizations, because there are far too many of them. We only mention the papers [14] and [4,5], in which the generalizations of exponential dichotomy are closest to the ones considered in the present paper.

We denote class of n-dimensional exponentially dichotomous system on the time half-line by $\mathcal{E}_n$ and the subclass of systems whose coefficient matrices are continuous of the half-line by $C\mathcal{E}_n$. In Definition 1.1, the positive constant factors c_1 and c_2 are the same for all solutions such that $x(0) \in L_-$ and $x(0) \in L_+$ respectively (or, in other words, estimates 1) ¨ 2) are uniform with respect to the constants c_1 and c_2 on L_- and L_+ respectively). In exactly the same way, estimates (1) and (2) are also uniform in the time variable; i.e., they hold for all $t \geqslant s$ starting from zero for all $x(0) \in L_-$ and $x(0) \in L_+$.

The question considered in the present paper is as follows. Is the condition that estimates (1) and (2) be uniform with respect to the constant factors or the time variable a necessary condition for the exponential dichotomy of system (1.1)? If yes, how strongly may the known properties of exponentially dichotomous systems change if these conditions are dropped?

In accordance with the preceding, let us introduce two more definitions.

Definition 1.2. A system in $\mathcal{M}_n$ is said to be *weakly exponentially dichotomous on the half-line* if there exist positive constants ν_1 and ν_2 and a decomposition of the space $\mathbb{R}^n$ of initial data (at $t = 0$) into a direct sum of subspaces L_- and L_+ (the case of zero dimension of the subspaces not being excluded) such that the solutions $x(\,\cdot\,)$ of the system satisfy the following two conditions:

(1′) if $x(0) \in L_-$, then $\|x(t)\| \leqslant c_1(x)e^{-\nu_1(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$;

(2′) if $x(0) \in L_+$, then $\|x(t)\| \geqslant c_2(x)e^{\nu_2(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$.

Here $c_1(x)$ and $c_2(x)$ are positive constants generally depending (as hinted in their notation) on the choice of the solution $x(\,\cdot\,)$.

Thus, the definition of weakly exponentially dichotomous systems differs from the definition of exponentially dichotomous systems only in that the condition for the estimates to be uniform in the respective constant factors is dropped.

Definition 1.3. A system in $\mathcal{M}_n$ is called *almost exponentially dichotomous on the half-line* if there exist positive constants c_1, c_2 and ν_1, ν_2 and a decomposition of the space $\mathbb{R}^n$ of initial data (at $t = 0$) into a direct sum of subspaces L_- and L_+ (the case of zero dimension of the subspaces not being excluded) such that the solutions $x(\,\cdot\,)$ of the system satisfy the following two conditions:

(1″) if $x(0) \in L_-$, then $\|x(t)\| \leqslant c_1 e^{-\nu_1(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant t_x$;

(2″) if $x(0) \in L_+$, then $\|x(t)\| \geqslant c_2 e^{\nu_2(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant t_x$.

Here t_x is a nonnegative number generally depending (as hinted in their notation) on the choice of the solution $x(\,\cdot\,)$.

Although conditions (1″) and (2″) imply the uniformity of the estimates in the constant factors c_1 and c_2, this is true not for all $t \geqslant s \geqslant 0$ (as the case for exponentially dichotomous systems) but only for $t \geqslant s$ greater than some t_x, which depends on the solution $x(\,\cdot\,)$.

The subspaces L_- and L_+ from Definitions 1.1–1.3 are called, respectively, *stable* and *unstable subspaces*, and the numbers $-\nu_1$ and ν_2 from Definitions 1.1–1.3 are called *dichotomy exponents*.

We denote the class of n-dimensional weakly exponentially dichotomous systems by $W\mathcal{E}_n$ and the class of n-dimensional almost exponentially dichotomous systems by $A\mathcal{E}_n$, with $CW\mathcal{E}_n$ and $CA\mathcal{E}_n$ being their respective subclasses consisting of systems whose coefficient matrices are continuous on the half-line. We have the relations $\mathcal{E}_1 = A\mathcal{E}_1 = W\mathcal{E}_1$. The class $W\mathcal{E}_n$ was introduced in paper [6], in which the authors used the constructions in [3] to prove that, in particular, for $n \geqslant 2$ one has the proper inclusion $\mathcal{E}_n \subset W\mathcal{E}_n$. Inclusion $A\mathcal{E}_n \subset W\mathcal{E}_n$ is obvious (that $A\mathcal{E}_n \neq W\mathcal{E}_n$ if $n \geqslant 2$ is stated below).

2 Main results

Lemma. *If the system is weakly exponentially dichotomous, then its stable subspaces L_- is uniquely determined and coincides with subspace $\mathcal{Z}_A$ of initial (at $t=0$) vectors of solutions vanishing at infinity, and the subspaces L_+ can be selected to be any subspaces complementing the subspace L_- to $\mathbb{R}^n$.*

A linear subspace of the space $\mathcal{X}(A)$ is called *lineal.* If L is a linear subspace of $\mathbb{R}^n$, then by $L(A;\,\cdot\,)$ we denote the lineal formed by solutions of the system $A\in\mathcal{M}_n$ with initial (at $t=0$) vectors from the subspace L; herewith $L(A;t)$ is a linear subspace of $\mathbb{R}^n$, formed by the vectors $x(t)$ of those solutions $x(\,\cdot\,)$, for which $x(0)\in L$. The lineals $L_-(A;\,\cdot\,)$ and $L_+(A;\,\cdot\,)$ of the system $A\in W\mathcal{E}_n$ are called *stable* and *unstable* lineals, respectively. For each $t\geqslant 0$, the subspaces $L_-(A;t)$ and $L_+(A;t)$ disjoint, i.e. $L_-(A;t)\cap L_+(A;t)=\{\mathbf{0}\}$.

By $A\mathcal{E}_n^m$ and $W\mathcal{E}_n^m$ we denote the subclasses of the classes $A\mathcal{E}_n$ and $W\mathcal{E}_n$, respectively, consisting of systems that have the dimension of their subspace L_- equal to m $(0\leqslant m\leqslant n)$, by $CA\mathcal{E}_n^m$ and $CW\mathcal{E}_n^m$ we denote those subclasses of the classes $A\mathcal{E}_n^m$ and $W\mathcal{E}_n^m$, respectively, whose coefficients are continuous. By lemma, classes $W\mathcal{E}_n^m$, $m=\overline{0,n}$, are pairwise disjoint ($W\mathcal{E}_n^{m_1}\cap W\mathcal{E}_n^{m_2}=\varnothing$ if $m_1\neq m_2$); i.e., $W\mathcal{E}_n=\bigsqcup\limits_{m=0}^{n}W\mathcal{E}_n^m$. Since $A\mathcal{E}_n^m=W\mathcal{E}_n^m\cap A\mathcal{E}_n$, it follows that the classes $A\mathcal{E}_n^m$, $m=\overline{0,n}$, are disjoint as well. Moreover, we have the obvious inclusions $\mathcal{E}_n^m\subset A\mathcal{E}_n^m\subset W\mathcal{E}_n^m$ and $C\mathcal{E}_n^m\subset CA\mathcal{E}_n^m\subset CW\mathcal{E}_n^m$ $m=\overline{0,n}$, where $\mathcal{E}_n^m$ is the subclass of $\mathcal{E}_n$ consisting of systems that have the dimension of their subspace L_- equal to m $(0\leqslant m\leqslant n)$, and $C\mathcal{E}_n^m$ is a subclass of the class $\mathcal{E}_n^m$, whose systems have continuous coefficients.

In [2] the following theorem was proved.

Theorem 2.1.

(1) *For $(n,m)=(1,0)$, $(n,m)=(1,1)$ and $(n,m)=(2,1)$, we have the relations $\mathcal{E}_n^m=A\mathcal{E}_n^m=W\mathcal{E}_n^m$.*

(2) *For the remaining pairs (n,m) of integer $n\in\mathbb{N}$ and $0\leqslant m\leqslant n$, the proper inclusions $\mathcal{E}_n^m\subset A\mathcal{E}_n^m\subset W\mathcal{E}_n^m$ hold and, moreover, there are the proper inclusions $C\mathcal{E}_n^m\subset CA\mathcal{E}_n^m\subset CW\mathcal{E}_n^m$.*

Since the definitions of the classes of weakly and almost dichotomous systems are quite close to the definition of the class of exponentially dichotomous systems, then, despite the result of Theorem 2.1, it seems plausible that the main properties of weakly exponentially dichotomous systems differ slightly from the properties of exponentially dichotomous systems. The report shows that this natural assumption is generally wrong.

Let us present the main properties of exponentially dichotomous systems.

(a) Recall that some property of points in a metric space is called *rough* in this space if the points possessing it form an open set. It is well known (see, for example, [9, p. 260]) that in the metric space $(\mathcal{M}_n,\operatorname{dist_u})$ with metric $\operatorname{dist_u}(A,B)=\sup\limits_{t\geqslant 0}\|A(t)-B(t)\|$ of uniform convergence on the half-line the property of a system to be exponentially dichotomous is rough, i.e. the set $\mathcal{E}_n$ is open in the space $(\mathcal{M}_n,\operatorname{dist_u})$. We also recall that the *edge* of a set in the topological space is called the set-theoretic difference between this set and its interior.

(b) If the system A is exponentially dichotomous, then the conjugate to it system $-A^{*}$ is also exponentially dichotomous; moreover, if $A\in\mathcal{E}_n^m$ and $-\nu_1$, ν_2 are dichotomy exponents of the system A, then $-A^{*}\in\mathcal{E}_n^{n-m}$ and $-\nu_2$, ν_1 are dichotomy exponents of the system $-A^{*}$. The above statement about systems, which are conjugate to exponentially dichotomous systems, follows easily, for example, from [15, p. 14, Theorem 1.1]. In particular, the class $\mathcal{E}_n$ of exponentially dichotomous systems is invariant under conjugation.

(c) For a system $A \in \mathcal{E}_n$, $n \geqslant 2$, let us consider its a stable lineal $L_-(A;\,\cdot\,)$ and an unstable lineal $L_+(A;\,\cdot\,)$ (we assume that both of them are different from the zero lineal). As noted above, for every $t \in \mathbb{R}_+$ the subspaces $L_-(A;t)$ and $L_+(A;t)$ are disjoint, so for every $t \geqslant 0$ the inequality $\angle\{L_-(A;t), L_+(A;t)\} > 0$ hold. It is well known (see, for example, [15, p. 10, Lemma 1.1]) that

$$\inf_{t \geqslant 0} \angle\bigl\{L_-(A;t), L_+(A;t)\bigr\} > 0, \tag{2.1}$$

i.e. for stable and unstable lineals of exponentially dichotomous systems, the angles between their corresponding subspaces are separated from zero on a half-line. Note that some strengthening of property (2.1) for exponentially dichotomous systems was established in [7].

Property (2.1) of finite-dimensional exponentially dichotomous systems is so important that when generalizing [9, p. 233–234], [4, p. 131] the concept of exponential dichotomy on linear differential systems in a Banach space, in order to preserve the main features of the theory, this property has to be included in the definition of exponentially dichotomous systems in Banach spaces as an independent condition.

The listed above properties of the class of exponentially dichotomous systems: roughness, invariance under the conjugation operation, and separation of the angles between the stable and any unstable lineals of solutions, do not hold for classes of weakly and almost exponentially dichotomous systems, as the following theorems show.

Theorem 2.2. *For any integer $n \geqslant 2$ in the metric space $\mathcal{M}_n$ with the topology of uniform convergence on the half-line, the interior of the set of weakly (almost) exponentially dichotomous systems coincides with the set of exponentially dichotomous systems, i.e.,* $\operatorname{int} W\mathcal{E}_n = \mathcal{E}_n$ *(respectively* $\operatorname{int} A\mathcal{E}_n = \mathcal{E}_n$*) for any $n \geqslant 2$.*

Theorem 2.2 and some simple considerations imply the following corollary.

Corollary. *In a metric space $\mathcal{M}_n$, $n \geqslant 2$, with the topology of uniform convergence on the half-line, the set $W\mathcal{E}_n$ (the set $A\mathcal{E}_n$) is neither open nor closed, all its points is limit points, and its edge* $\operatorname{ed} W\mathcal{E}_n$ $(\operatorname{ed} A\mathcal{E}_n)$ *are exactly weakly (almost) exponentially dichotomous systems that are not exponentially dichotomous.*

This corollary, in particular, shows that the properties of a system to be weakly or almost exponentially dichotomous are not rough.

Theorem 2.2 and the corollary remain valid if the space $\mathcal{M}_n$ in them is replaced by its subspace $C\mathcal{M}_n$, and the subsets $W\mathcal{E}_n$, $A\mathcal{E}_n$, and $\mathcal{E}_n$ by the subsets $CW\mathcal{E}_n$, $CA\mathcal{E}_n$, and $C\mathcal{E}_n$, respectively.

The non-invariance of the classes $W\mathcal{E}_n$ and $A\mathcal{E}_n$, if $n \geqslant 2$, under conjugation is stated by the following theorem.

Theorem 2.3. *For any $n \geqslant 2$ there exists a continuous n-dimensional a weakly (almost) exponentially dichotomous system such that its conjugate system is not weakly (almost) exponentially dichotomous.*

In the general case, the non-separation from zero of the angle between the stable $L_-(\,\cdot\,)$ and some unstable $L_+(\,\cdot\,)$ lineals of a weakly (almost) exponentially dichotomous system is established by the following theorem.

Theorem 2.4. *For any integer $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$ in the class $CA\mathcal{E}_n^m$ there exists a system such that the angle between its stable lineal $L_-(\,\cdot\,)$ and some unstable lineal $L_+(\,\cdot\,)$ is not separated from zero, i.e.* $\inf\limits_{t \geqslant 0} \angle\bigl(L_-(t), L_+(t)\bigr) = 0.$

Note that the restrictions $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$ in the statement of Theorem 2.4 are essential: if m is equal to 0 or n, then one of the lineals $L_-(\,\cdot\,)$ or $L_+(\,\cdot\,)$ is zero and the angle $\angle\big(L_-(t), L_+(t)\big)$ is undefined; if $n=2$, then for $m=1$ the system is exponentially dichotomous, which means that Theorem 2.4 is not true for it.

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