On Extensibility and Asymptotics of Solutions to the Riccati Equation with Real Roots of its Right Part

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Consider the Riccati equation

$$y' = P(x) + Q(x)y + y^2, \tag{1}$$

where $P(x)$ and $Q(x)$ are continuous functions bounded on $(-\infty;\infty)$. Suppose the equation

$$y^2 + Q(x)y + P(x) = 0$$

has real bounded roots $\alpha_1(x) \in C^1(-\infty,+\infty)$ and $\alpha_2(x) \in C^1(-\infty,+\infty)$. So, equation (1) can be written as

$$y'(x) = (y(x) - \alpha_1(x))(y(x) - \alpha_2(x)). \tag{2}$$

Thus we have

$$Q^2(x) - 4P(x) \geqslant 0.$$

Suppose that either $\alpha_2(x) > \alpha_1(x)$, $x \in (-\infty,+\infty)$, or $\alpha_2(x) = \alpha_1(x)$, $x \in (-\infty,+\infty)$, that $\alpha_1(x)$ and $\alpha_2(x)$ are bounded C^1 functions on $(-\infty,+\infty)$.

We define a function $Y_0(x)$ by

$$Y_0(x) := \Big[\frac{(\alpha_1(x) - \alpha_2(x))^2}{4} + \frac{(\alpha_1(x) + \alpha_2(x))'}{2}\Big].$$

Lemma 1 ([4, Lemma 4.1])**.** *Suppose $x_0 < \omega \le +\infty$. Then there exist $S_* \in [x_0,\omega)$ and a solution $y_*(x)$ to equation* (2) *defined on (S_*,ω) such that any solution $y(t)$ defined on (S,ω) satisfies $S \ge S_*$ and $y(x) \le y_*(x)$ for all $x \in (S,\omega)$.*

Hereafter the solution $y_*(x)$ from the last lemma is called a **principal solution**.

Definition 1 ([5])**.** The functions α_1 and α_2 in equation (2) are said to satisfy the **stabilization conditions** if there exist finite limits

$$\lim_{x\to\pm\infty} \alpha_j(x) =: \alpha_{j,\pm} \in \mathbb{R}, \;\; j = 1, 2. \tag{3}$$

Definition 2 ([5])**.** The functions α_1 and α_2 are said to satisfy the **monotone stabilization conditions** if there exists $A > 0$ such that

$$\alpha_1'(x) \neq 0, \;\; \alpha_2'(x) \neq 0 \;\text{ for all }\; x \notin [-A, A]. \tag{4}$$

Definition 3 ([5])**.** A solution $y(x)$ to equation (2) is called **stabilizing** if there exist finite limits

$$\lim_{x\to\pm\infty} y(x) =: y_\pm \in \mathbb{R}.$$

Theorem 1. *Suppose*

$$Q'(x) < \frac{Q^2(x)}{2} - 2P(x) \ \textit{ for all } \ x \geq x_0.$$

Then any solution $y(x)$ *to equation* (2) *with* $y(x_0) \leq -\frac{Q(x_0)}{2}$ *satisfies also the condition*

$$y(x) < -\frac{Q(x)}{2} \ \textit{ for all } \ x > x_0.$$

Corollary 1. *Suppose that* $\alpha_1(x) = \alpha_2(x) = \alpha(x)$ *for all* $x \in (-\infty, +\infty)$ *and* $\alpha'(x) > 0$ *for all* $x \geq x_0$. *Then any solution* $y(x)$ *to equation* (2) *with* $y(x_0) \leq \alpha(x_0)$ *satisfies also the condition* $y(x) < \alpha(x)$ *for all* $x > x_0$.

Theorem 2. *Any solution to equation* (2) *defined at some* $x_0 \in \mathbb{R}$ *is bounded below to the right of* x_0.

Theorem 3. *Suppose there exists a constant* M *such that* $\alpha_1(x) \leq \alpha_2(x) \leq M$ *for all* $x \geq x_0$. *Then any solution* $y(x)$ *to equation* (2) *with* $y_0 = y(x_0) > M$ *monotonically increases to the right of* x_0 *and*

$$\lim_{x \to \overline{x}} y(x) = +\infty \ \textit{ with } x_0 < \overline{x} < x_0 + \frac{1}{y_0 - M}\,.$$

Note that in the particular case $\alpha_2(x) = \alpha_1(x)$ on $(-\infty, +\infty)$, Theorem 3 yields the first statement of Theorem 5.5 from [2].

Now by using the substitutions $\widehat{x} = -x$, $\widehat{y}(\widehat{x}) = -y(-\widehat{x})$ we transform equation (2) to the form

$$\frac{d}{d\widehat{x}}\,\widehat{y}(\widehat{x}) = \big(\widehat{y}(\widehat{x}) - \widehat{\alpha}_1(\widehat{x})\big)\big(\widehat{y} - \widehat{\alpha}_2(\widehat{x})\big),$$

where $\widehat{\alpha}_1(\widehat{x}) = -\alpha_1(x)$, $\widehat{\alpha}_2(\widehat{x}) = -\alpha_2(x)$. Thus, we can obtain analogues of Theorems 1–3 and their corollaries for the case $x \leq x_0$. In particular, the following theorem is an analogue of Theorem 3.

Theorem 3′. *If there exists a constant* m *such that* $\alpha_2(x) \geq \alpha_1(x) \geq m$ *for all* $x \leq x_0$, *then every solution* $y(x)$ *to* (2) *with* $y_0 = y(x_0) < m$ *is monotonic for* $x \geq x_0$ *and*

$$\lim_{x \to \overline{x}} y(x) = -\infty, \quad \textit{where } \ x_0 > \overline{x} > x_0 - \frac{1}{m - y_0}\,.$$

Obtained Theorems 1–3 and 3′ complement results of [2]. We used results of [4,5] and the proof of Lemma 7.1 ([3, p. 365]) to obtain the following theorems.

Theorem 4. *Let* $y_3(x) < y_2(x) < y_1(x)$ *be different solutions to* (2) *defined at a point* x_0 *and* y_1 *be extensible on* $[x_0, +\infty)$. *Then* y_2 *and* y_3 *are also extensible on* $[x_0, +\infty)$ *with the following properties:*

1) *The ratio* $\frac{y_1(x)-y_3(x)}{y_1(x)-y_2(x)}$ *is monotonically decreasing on* $[x_0, +\infty)$;

2) *There exists a finite limit* $\lim\limits_{x \to +\infty} \frac{y_1(x)-y_3(x)}{y_1(x)-y_2(x)}$;

3) *If* $y_1(x)$ *is a principal solution for the interval* $(x_0, +\infty)$, *then the above limit equals* 1.

Theorem 5. *Let* $y_1(x)$, $y_2(x)$ *be two different solutions to* (2) *defined on* $[x_0, +\infty)$. *Let both of them have different finite limits as* $x \to +\infty$. *Then every solution to (2) defined on* $[x_0, +\infty)$ *has a finite limit as* $x \to +\infty$.

Theorem 6. *Let $y_1(x) > y_2(x)$ be two different solutions to* (2) *defined on $[x_0, +\infty)$. Let both of them have finite limits as $x \to +\infty$. Then every solution to* (2) *defined at the point x_0 with $y(x_0) \le y_1(x_0)$ is extensible on $[x_0, +\infty)$ and has a finite limit as $x \to +\infty$.*

Theorem 7. *Let* (2) *have solutions defined on $[x_0, +\infty)$. Let $y_1(x) = y_*(x)$ be the principal solution for the interval $(x_0, +\infty)$. Let $y_1(x)$ and another solution $y_2(x) < y_1(x)$ have finite limits as $x \to +\infty$. Then every solution to* (2) *defined on $[x_0, +\infty)$ and different from $y_*(x)$ has a finite limit as $x \to +\infty$. This limit is equal to the limit of $y_2(x)$ as $x \to +\infty$.*

Further we assume that the functions $\alpha_1(x)$ and $\alpha_2(x)$ are bounded and satisfy (3), (4), and $\alpha_2(x) > \alpha_1(x)$, $x \in (-\infty, +\infty)$. As shown in [5], in this case all bounded solutions are stabilizing (and vice versa), all stabilizing solutions are monotonically stabilizing and y_- equals $\alpha_{1,-}$ or $\alpha_{2,-}$, while y_+ equals $\alpha_{1,+}$ or $\alpha_{2,+}$.

According to [5], all stabilizing solutions to (2) are divided into four types:

type I: $y_- = \alpha_{1,-},\ y_+ = \alpha_{1,+}$.

type II: $y_- = \alpha_{2,-},\ y_+ = \alpha_{1,+}$.

type III: $y_- = \alpha_{2,-},\ y_+ = \alpha_{2,+}$.

type IV: $y_- = \alpha_{1,-},\ y_+ = \alpha_{2,+}$.

Theorem 8. *Suppose $\alpha_{1,+} \ne \alpha_{2,+}$, $\alpha_{1,-} \ne \alpha_{2,-}$, and $Y_0(x) \le 0$ on $\mathbb{R} \setminus [a, b]$. Then all solutions to* (2) *are not stabilizing.*

The last theorem complements Theorem 3.4 from [5].

Theorem 9. *Suppose that $\alpha_{1,+} \ne \alpha_{2,+}$, $\alpha_{1,-} \ne \alpha_{2,-}$, and equation* (2) *has a stabilizing solution of type* II. *Then there exist a unique solution of type* I *and a unique solution of type* III. *Denote them by y_I and y_{III}, respectively. Let $y(x)$ be a solution to* (2). *Then:*

- *if $y_I < y < y_{III}$, then $y(x)$ is a stabilizing solution of type* II;
- *if $y > y_{III}$, then there exists $x^* \in \mathbb{R}$ such that $y(x)$ is extensible on the interval $(-\infty, x^*)$ and*
$$\lim_{x \to -\infty} y(x) = \alpha_{2,-}, \quad \lim_{x \to x^*-0} y(x) = +\infty;$$
- *if $y < y_I$, then there exists $x^* \in \mathbb{R}$ such that $y(x)$ is extensible on the interval $(x^*, +\infty)$ and*
$$\lim_{x \to +\infty} y(x) = \alpha_{1,+}, \quad \lim_{x \to x^*+0} y(x) = -\infty.$$

Theorem 10. *Suppose $\alpha_{1,+} \ne \alpha_{2,+}$, $\alpha_{1,-} \ne \alpha_{2,-}$. Then the following conditions are equivalent.*

1) *There exist stabilizing solutions to* (2) *of type* I *and of type* III.
2) *There exist a unique stabilizing solution to* (2) *of type* I *and a unique stabilizing solution to* (2) *of type* III.
3) *There exists a stabilizing solution to* (2) *of type* II.

Theorem 11. *Suppose $\alpha_{1,+} \ne \alpha_{2,+}$, $\alpha_{1,-} \ne \alpha_{2,-}$. Then exactly one of the following statements is true:*

1) *There exists a stabilizing solution to* (2) *of type* II.
2) *There exist a stabilizing solutions to* (2) *of type* I *and a unique stabilizing solution of type* IV.
3) *There exist a stabilizing solution to* (2) *of type* III *and a unique stabilizing solution of type* IV.
4) *All stabilizing solutions, if any, are stabilizing solutions of type* IV.

Theorems 8–11 complement Theorems 2.1–2.4 from [5].

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