Double Minimum Control Problem for a Parabolic Equation

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1 Introduction

We consider an extremum problem with weighted integral cost functional for the following parabolic mixed problem

$$u_t = (a(x,t)u_x)_x + b(x,t)u_x + h(x,t)u, \ \ (x,t) \in Q_T = (0,1) \times (0,T), \ \ T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \ \ 0 < t < T, \tag{1.2}$$

$$u(x,0) = \xi(x), \ \ 0 < x < 1, \tag{1.3}$$

where the real functions a, b and h are smooth in $\overline{Q}_T$, $0 < a_0 \le a(x,t) \le a_1 < \infty$, $\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. Here $W_2^1(0,T)$ is the Sobolev space of weakly differentiable functions with the norm

$$\|u\|^2_{W_2^1(0,T)} = \int\limits_0^T ({u'}^2 + u^2)\,dt.$$

We study the control problem with pointwise observation: by controlling the temperature φ at the left end of the segment (the functions ψ and ξ are assumed to be fixed), we try to make at some point $x_0 \in (0,1)$ the temperature $u(x_0,t)$ close to the given function $z(t)$ over the entire time interval $(0,T)$. This problem arises in the model of climate control in industrial greenhouses [1, 6]. Note that extremal problems for parabolic equations were considered in [11, 15, 17, 18] (as usual, problems with final or distributed observation). But the results and methods of investigation are not similar to our methods. The proposed paper develops and generalizes the authors' results of [1–8]. Here we study a more general equation with a variable diffusion coefficient a, a convection coefficient b, and a potential h called the depletion potential. We state a problem of double minimization to our functional obtain by finding first minimum of the functional in some class of control functions and iterated minimum by weight. For such problem we prove the existence of a pair of minimizers.

As well as in [13, p. 6], we denote by $V_2^{1,0}(Q_T)$ the Banach space of functions $u \in W_2^{1,0}(Q_T)$ with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0\le t\le T} \|u(\,\cdot\,,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)}$$

such that $t \mapsto u(\,\cdot\,,t)$ is a continuous mapping from $[0,T]$ to $L_2(0,1)$. Let $\widetilde{W}_2^1(Q_T)$ be set of all functions $\eta \in W_2^1(Q_T)$ satisfying the conditions $\eta(\,\cdot\,,T) = 0$, $\eta(0,\,\cdot\,) = 0$.

Definition 1.1. A function $u \in V_2^{1,0}(Q_T)$, satisfying the condition $u(0,t) = \varphi(t)$ and the equality

$$\int\limits_{Q_T} (a(x,t)u_x\eta_x - b(x,t)u_x\eta - h(x,t)u\eta - u\eta_t)\,dx\,dt$$

$$= \int\limits_0^1 \xi(x)\eta(x,0)\,dx + \int\limits_0^T a(1,t)\psi(t)\,\eta(1,t)\,dt \qquad (1.4)$$

for all $\eta \in \widetilde{W}_2^1(Q_T)$, is called a weak solution to problem (1.1)–(1.3).

2 Main Results

Theorem 2.1. *The problem* (1.1)–(1.3) *has a unique weak solution $u \in V_2^{1,0}(Q_T)$, which satisfies the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_1\Big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)}\Big) \qquad (2.1)$$

with some constant C_1 independent of φ, ψ and ξ.

Corollary 2.1. *The solution u to problem* (1.1)–(1.3) *continuously depends on the triple* (ξ, φ, ψ) *from $L_2(0,1) \times W_2^1(0,T) \times W_2^1(0,T)$.*

We consider a set of control functions $\varphi \in W_2^1(0,T)$ and a set of objective functions $z \in L_2(0,T)$. These sets are denoted by Φ and Z. Hereafter we suppose that Φ is a non-empty, closed, convex, and bounded set. Consider the weighted integral cost functional

$$J[z,\rho,\varphi] = \int\limits_0^T (u_\varphi(x_0,t) - z(t))^2\rho(t)\,dt, \quad x_0 \in (0,1), \quad \varphi \in \Phi, \quad z \in Z,$$

where $u_\varphi \in V_2^{1,0}(Q_T)$ is the solution to problem (1.1)–(1.3) with the given control function φ. Here $\rho \in L_\infty(0,T)$ is a real-valued weight function such that $\operatorname{ess}\inf\limits_{t\in(0,T)} \rho(t) > 0$. Assuming the functions z and ρ to be fixed, consider the minimization problem of finding

$$m[z,\rho,\Phi] = \inf_{\varphi\in\Phi} J[z,\rho,\varphi].$$

Theorem 2.2 ([5,8,9])**.** *For any $z \in L_2(0,T)$ there exists a unique function $\varphi_0 \in \Phi$ such that*

$$m[z,\rho,\Phi] = J[z,\rho,\varphi_0].$$

Take $\widetilde{\widetilde{\rho}} > \widetilde{\rho} > 0$, we consider the set $P \subset L_\infty(0,T)$ of all weight functions ρ with

$$\operatorname{ess}\inf_{t\in(0,T)} \rho(t) \geqslant \widetilde{\rho}, \quad \operatorname{ess}\sup_{t\in(0,T)} \rho(t) \leqslant \widetilde{\widetilde{\rho}}.$$

Let us state for some subset $\widetilde{P} \subset P$ the double minimum problem

$$\mu[z,\widetilde{P},\Phi] = \inf_{\rho\in\widetilde{P}} m[z,\rho,\Phi].$$

Definition 2.1 ([12])**.** Let X be a Banach space. The set $Y \subset X^*$ is called a regularly convex if for any $y \notin Y$ there exists an element $x_0 \in X$ such that

$$\sup_{f\in Y} f(x_0) < y(x_0).$$

Theorem 2.3. *Let the set $\widetilde{P}$ be regularly convex in $L_\infty(0,T)$. Then for all $z \in L_2(0,T)$ there exist functions $\rho_0 \in \widetilde{P}$ and $\varphi_0 \in \Phi$ such that*

$$\mu[z,\widetilde{P},\Phi] = J[z,\rho_0,\varphi_0].$$

3 Proofs

At first we establish the following generalization of the classical maximum principle (see [13, Ch. 3, Par. 7]).

Lemma 3.1. *If $u \in V_2^{1,0}(Q_T)$ is a solution to the problem*

$$\begin{gathered} u_t = (a(x,t)u_x)_x + b(x,t)u_x + h(x,t)u, \ \ (x,t) \in Q_T, \\ u(0,t) = \varphi(t), \quad u_x(1,t) = 0, \ \ 0 < x < 1, \ \ t > 0, \\ u(x,0) = 0, \ \ 0 < x < 1, \end{gathered} \tag{3.1}$$

then the inequality

$$\operatorname{ess}\sup_{(x,t)\in Q_T} |u(x,t)| \le C_2 \sup_{t\in[0,T]} |\varphi(t)| \tag{3.2}$$

holds with a constant $C_2 > 0$ depending only on the coefficients of equation (3.1).

Also we will use the following statements to prove Theorem 2.3.

Theorem 3.1 ([12, Theorem 10])**.** *Let X be a separable Banach space. Then a set $Y \subset X^*$ is regularly convex if and only if it is convex and $*$-weakly closed.*

Theorem 3.2 ([10, Ch. 8, §7])**.** *For any bounded sequence $(\rho_k)_{k\in\mathbb{N}}$ in $L_\infty(0,T)$ there exist a subsequence $(\rho_{k_j})_{j\in\mathbb{N}}$ and a function $\rho_0 \in L_\infty(0,T)$ such that*

$$\lim_{j\to+\infty} \int_0^T \rho_{k_j}(t)\zeta(t)\,dt = \int_0^T \rho_0(t)\zeta(t)\,dt$$

for any function $\zeta \in L_1(0,T)$.

Proof of Theorem 2.3. Put $d = \mu[z,\widetilde{P},\Phi]$. Then there exists a sequence of weight functions $\rho_1,\rho_2,\ldots \in \widetilde{P}$ such that

$$m[z,\rho_k,\Phi] \to d, \ \ k \to \infty. \tag{3.3}$$

So, by (3.3) and Theorem 2.2 there exists a sequence of control functions $\varphi_1,\varphi_2,\dots \in \Phi$ satisfying

$$J[z,\rho_k,\varphi_k] = m[z,\rho_k,\Phi] \to d, \ \ k \to \infty.$$

The functions φ_k belong to Φ, so, the sequence of norms $\|\varphi_k\|_{W_2^1(0,T)}$ is bounded due to boundedness of the set Φ. Therefore, there exists a subsequence $(\varphi_{k_j})_{j\in\mathbb{N}}$ converging weakly in $W_2^1(0,T)$ to some function $\varphi_0 \in \Phi$ due to closeness of the set Φ. Now, by compact embedding of $W_2^1(0,T)$ into $C([0,T])$, the sequence $(\varphi_{k_j})_{j\in\mathbb{N}}$ converges to φ_0 by norm of $C([0,T])$:

$$\|\varphi_{k_j} - \varphi_0\|_{C([0,T])} \to 0, \ \ j \to \infty. \tag{3.4}$$

Further we write φ_k instead if φ_{k_j}.

The next step is to study behavior of the sequence of solutions $u_k = u_{\varphi_k}$, $k = 1, 2, \dots$, to problem (1.1)–(1.3) in the space $W_2^{1,0}(Q_T)$. Functions u_k are solutions of the following mixed problems:

$$u_{kt} = (a(x,t)u_{kx})_x + b(x,t)u_{kx} + h(x,t)u_k, \quad (x,t) \in Q_T,$$
$$u_k(0,t) = \varphi_k(t), \quad u_{kx}(1,t) = \psi(t), \quad 0 < t < T,$$
$$u_k(x,0) = \xi(x), \quad 0 < x < 1.$$

Functions u_{φ_k} satisfy the condition $u_{\varphi_k}(0,t) = \varphi_k(t)$ and by (1.4) the equalities

$$\int\limits_{Q_T} (a(x,t)u_{\varphi_k x}\eta_x - b(x,t)u_{\varphi_k x}\eta - h(x,t)u_{\varphi_k}\eta - u_{\varphi_k}\eta_t)\, dx\, dt$$
$$= \int\limits_0^1 \xi(x)\eta(x,0)\, dx + \int\limits_0^T a(1,t)\psi(t)\,\eta(1,t)\, dt \tag{3.5}$$

for all $\eta \in \widetilde{W_2^1}(Q_T)$. It follows from (2.1) that

$$\|u_{\varphi_k}\|_{W_2^{1,0}(Q_T)} \le C_3\|u_{\varphi_k}\|_{V_2^{1,0}(Q_T)} \le C_4\Big(\|\varphi_k\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)}\Big) \le C_5$$

with a constant C_5 independent of k. So, there exists a subsequence $u_{\varphi_{k_j}}$ (we denote it by u_j) such that $u_j \to u_0$, $j \to \infty$, weakly for some $u_0 \in W_2^{1,0}(Q_T)$. From (3.5) and the weak convergence of the sequence u_j in $W_2^{1,0}(Q_T)$, it follows that the weak limit (the function u_0) satisfies equality (1.4) for all $\eta \in \widetilde{W_2^1}(Q_T)$. Now, we prove that $u_0|_{x=0} = \varphi_0$. By the Banach–Saks theorem [16, Ch. 2, Sec. 38] we have a subsequence (we denote it by u_j too) such that

$$\|\widehat{u}_k - u_0\|_{W_2^1(Q_T)} \to 0, \quad k \to \infty, \quad \widehat{u}_k = \frac{1}{k}\sum_{j=1}^k u_j. \tag{3.6}$$

Therefore,

$$\|\widehat{u}_k(0,t) - u_0(0,t)\|_{L_2(0,T)} \le C_6\|\widehat{u}_k - u_0\|_{W_2^1(Q_T)} \to 0, \quad k \to \infty. \tag{3.7}$$

But it follows from (3.6), (3.7) that in the $L_2(0,T)$ space we have

$$u_0(0,\,\cdot\,) = s - \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^k \varphi_j(\,\cdot\,) = w - \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^k \varphi_j(\,\cdot\,) = w - \lim_{k\to\infty} \varphi_k(\,\cdot\,) = \varphi_0(\,\cdot\,).$$

(If φ_k converges to φ_0 weakly in $W_2^1(0,T)$, then it converges weakly to φ_0 in $L_2(0,T)$ too.) So, the limit function u satisfies $u_0|_{x=0} = \varphi_0$. It means that u is a solution to problem (1.1)–(1.3) with the control function $\varphi = \varphi_0$. Let $v_k = u_{\varphi_k} - u_{\varphi_0}$. Functions v_k are solutions to the following mixed problems:

$$v_{kt} = (a(x,t)v_{kx})_x + b(x,t)v_{kx} + h(x,t)v_k, \quad (x,t) \in Q_T,$$
$$v_k(0,t) = \varphi_k(t) - \varphi_0(t), \quad v_{kx}(1,t) = 0, \quad 0 < t < T,$$
$$v_k(x,0) = 0, \quad 0 < x < 1.$$

By inequalities (3.2) and (3.4) we obtain that

$$\|v_k(x_0,t)\|_{L_2(0,T)} \le \sqrt{T}\,\|v_k(x_0,t)\|_{C([0,T])} \to 0, \quad k \to \infty.$$

So, the sequence of functions $\{(u_k(x_0,\,\cdot\,)-z(\,\cdot\,))^2\}_{k=1}^{\infty}$ converges by norm in the $L_1(0,T)$ space to the function $(u_0(x_0,\,\cdot\,)-z(\,\cdot\,))^2$. Now, by Theorem 3.1 we can extract from the minimizing sequence of weight functions $\rho_k(t)$ a subsequence (we will denote it also $\rho_k(t)$) that *-weakly converges in $L_\infty(0,T)$ to some $\rho_0\in\widetilde{P}$. Combining this with Theorem 3.2, we obtain the following relation:

$$\mu[z,\widetilde{P},\Phi]=\lim_{k\to\infty}\int\limits_0^T(u_{\varphi_k}(x_0,t)-z(t))^2\rho_k(t)\,dt=\int\limits_0^T(u_{\varphi_0}(x_0,t)-z(t))^2\rho_0(t)\,dt=J[z,\rho_0,\varphi_0].$$

Proof of Theorem 2.3 is completed. □

Acknowledgements

The reported study was partially supported by RSF, project # 20-11-20272.

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