Application of the Fractional Power Series to Solving Some Fractional Emden–Fowler Type Equations

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1 Introduction

Fractional differential equations have already proved to be valuable tools to the modelling of many physical phenomena [2, 4, 5, 8]. There are many techniques for solving fractional differential equations, in particular, RFPS method (residual fractional power series), which allows us to obtain solutions to initial value problems for Emden–Fowler type equations in the form of fractional power series [7]. These equations have many applications in the fields of radioactivity cooling and in the mean-field treatment of a phase transition in critical adsorption, kinetics of combustion or reactants concentration in chemical reactor and isothermal gas spheres and thermionic currents [1, 9].

2 Problem statement

Definition 2.1 ([4])**.** For $\mu \in \mathbb{R}$ the space C_μ is the space of functions f given on the half-axis $\mathbb{R}_+ \equiv [0, +\infty)$ and represented in the form $f = x^p f_1$ for some $p > \mu$, where the function f_1 is continuous on $\mathbb{R}_+$:

$$C_\mu = \Big\{ f : \;\; f = x^p f_1, \;\; f_1` \in C(\mathbb{R}_+) \;\text{ for some }\; p > \mu \in \mathbb{R} \Big\}.$$

Similarly, the space C_μ^n is the space of functions f given on the half-axis $\mathbb{R}_+$ such that $f^{(n)} \in C_\mu$.

Definition 2.2. For given $x_0 \geq 0$ *the* α *order Caputo fractional derivative* of function $f \in C_{-1}^n$ such that $f^{(n)}|_{x=x_0} = 0$, where $\alpha \in [n, n+1)$, $n \in \mathbb{N}$, is defined by

$${}_{x_0}^{\;C}D_x^\alpha f \equiv \frac{1}{\Gamma(n - \alpha + 1)} \int\limits_{x_0}^{x} (x-t)^{n-\alpha} f^{(n+1)}(t)\,\mathrm{d}t \;\text{ or, respectively, }\; {}_{x_0}^{\;C}D_x^n f \equiv f^{(n)}(x).$$

Definition 2.3. For given $\alpha \geq 0$ *the fractional power series* (FPS) around the center $x_0 \in \mathbb{R}$ is a functional series of the following form:

$$\sum_{n=0}^{+\infty} c_n (x - x_0)^{n\alpha}, \;\; x \geq x_0.$$

Properties of FPS are presented in [3]. Let $\alpha \in (1/2, 1]$, and $D_x^\alpha \equiv {}_0^C D_x^\alpha$. We consider the following initial value problem (IVP):

$$D_x^{2\alpha} u + \frac{a}{x^\alpha} D_x^\alpha u + s(x) g(u) = h(x), \;\; x > 0, \;\; u(0) = \widehat{u}_0, \;\; D_x^\alpha u\big|_{x=0} = 0, \tag{1}$$

where

$$s(x)\equiv\sum_{n=0}^{+\infty}s_n\,\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\,,\quad h(x)\equiv\sum_{n=0}^{+\infty}h_n\,\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\,,\quad g(u)\equiv\sum_{k=0}^{K}a_ku^k,\ \ K\in\mathbb{N}.$$

Using FPS the solution to IVP (1) can be written as

$$u(x)=\sum_{n=0}^{+\infty}u_n\,\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\quad\bigg(U_N(x)\equiv\sum_{n=0}^{N}u_n\,\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\bigg). \tag{2}$$

3 Main results

Theorem. *If the solution to IVP* (1) *is sought in the form of series* (2)*, then the following equalities hold:* $u_0=\widehat{u}_0$, $u_1=0$ *and*

$$u_N\bigg(1+\frac{a\Gamma(1+(N-2)\alpha)}{\Gamma(1+(N-1)\alpha)}\bigg)=h_{N-2}-D_x^{(N-2)\alpha}\bigg(\sum_{n=0}^{N}s_n\,\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\sum_{k=0}^{K}a_kU_N^k(x)\bigg)\bigg|_{x=0},\quad N\geq 2.$$

Corollary 3.1. *If* $s(x)\equiv s\in\mathbb{R}$, $h(x)\equiv 0$ *and* $g(u)\equiv u$, *then the solution to problem* (1) *is given in the form of the following series*

$$u(x)=\widehat{u}_0+\sum_{n=1}^{+\infty}(-1)^ns^n\widehat{u}_0\bigg(\prod_{k=1}^{n}\frac{\Gamma(1+(2k-1)\alpha)}{\Gamma(1+(2k-1)\alpha)+a\Gamma(1+2(k-1)\alpha)}\bigg)\frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)}\,,$$

which converges absolutely and uniformly for $x\geq 0$.

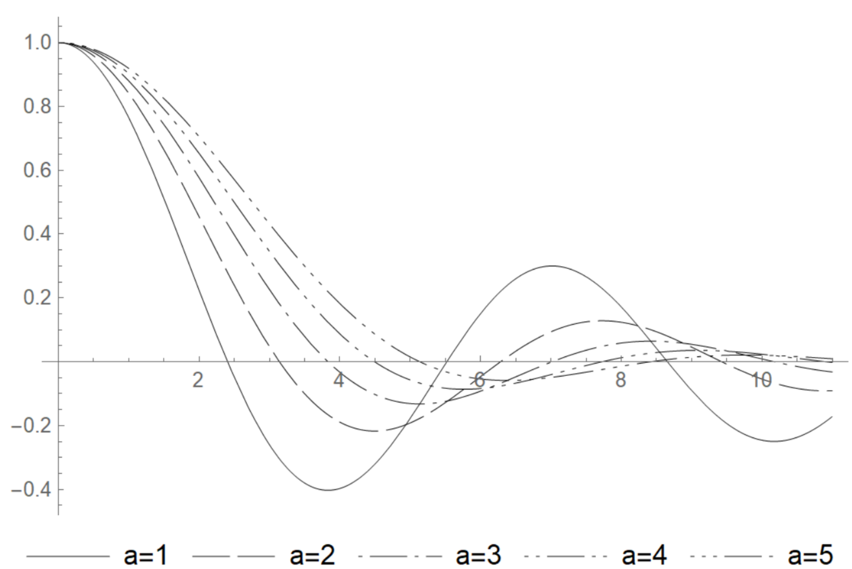

Figure 1. Graphs of the solutions to IVP (1) in case $s(x)\equiv s\in\mathbb{R}$, $h(x)\equiv 0$, $g(u)\equiv u$ and $\alpha=\widehat{u}_0=s=1$ and various values of a.

Under the conditions of Corollary 3.1 and in case of integer order differential operator ($\alpha=1$) we obtain the solutions to IVP (1), if $\widehat{u}_0=s(x)\equiv 1$ and $a=\overline{1,5}$:

$$u(x)=J_0(x),\ \frac{\sin x}{x},\ \frac{2J_1(x)}{x},\ \frac{3\sin x-3x\cos x}{x^3},\ \frac{8J_2(x)}{x^2}\,,$$

where $J_a(x)$ are Bessel functions of the first kind. It is noteworthy that in [6] in case $a = 2$ the same solution was obtained by using fractional differential transformation method (FDT).

Corollary 3.2. *If* $s(x) \equiv x^\alpha$ *and*

$$h(x) \equiv \Gamma(1+2\alpha) + \frac{a\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} + x^\alpha(\widehat{u}_0 + x^{2\alpha})^k, \quad g(u) \equiv u^k,$$

where $k \in \mathbb{N}$*, then IVP* (1) *has a solution* $u(x) = \widehat{u}_0 + x^{2\alpha}$*.*

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