On Initial-Periodic Type Problems for Three-Dimensional Linear Hyperbolic System

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In the rectangular box $\Omega=[0,\omega_1]\times[0,\omega_2]\times[0,\omega_3]$ for the linear hyperbolic system

$$u^{(\mathbf{1})}=\sum_{\boldsymbol{\alpha}<\mathbf{1}}P_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}) \tag{1}$$

consider the initial-periodic conditions

$$\begin{gathered}u(0,x_2,x_3)=\varphi_1(x_2,x_3),\quad u^{(1,0,0)}(x_1,0,x_3)=\varphi_2^{(1,0)}(x_1,x_3)\\ u(x_1,x_2,x_3+\omega_3)=u(x_1,x_2,x_3)\end{gathered} \tag{2}$$

and

$$\begin{gathered}u(0,x_2,x_3)=\varphi(x_2,x_3),\\ u(x_1,x_2+\omega_2,x_3)=u(x_1,x_2,x_3),\quad u(x_1,x_2,x_3+\omega_3)=u(x_1,x_2,x_3)\end{gathered} \tag{3}$$

Here $\mathbf{x}=(x_1,x_2,x_3)$, $\mathbf{1}=(1,1,1)$ and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x})=\frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\,,$$

$P_{\boldsymbol{\alpha}}\in C(\Omega;\mathbb{R}^{n\times n})$ $(\boldsymbol{\alpha}<\mathbf{1})$, $q\in C(\Omega;\mathbb{R}^n)$, $\varphi_1\in C^{1,1}(\Omega_{23})$, $\varphi_2\in C^{1,1}(\Omega_{13})$, $\Omega_{23}=[0,\omega_2]\times[0,\omega_3]$ and $\Omega_{13}=[0,\omega_1]\times[0,\omega_3]$.

Throughout the paper the following g notations will be used:

$\mathbf{0}=(0,0,0)$, $\mathbf{1}=(1,1,1)$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)<\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\Longleftrightarrow\alpha_i\le\beta_i$ $(i=1,2,3)$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\le\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\Longleftrightarrow\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\|=|\alpha_1|+|\alpha_2|+|\alpha_3|$.

Let $\mathbf{m}=(m_1,m_2,m_3)$ be a multi-index. By $C^{\mathbf{m}}(\Omega;\mathbb{R}^n)$ denote the Banach space of vector functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)}=\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

By a solution of problem (1), (2) (problem (1), (3)) we understand a classical solution, i.e., a vector-function $u\in C^{\mathbf{1}}(\Omega;\mathbb{R}^n)$ satisfying system (1) and boundary conditions (2) (system (1) and boundary conditions (3)) everywhere in Ω.

Along with system (1) consider its corresponding homogeneous system

$$u^{(\mathbf{1})}=\sum_{\boldsymbol{\alpha}<\mathbf{1}}P_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{1_0}$$

and the following boundary value problems

$$\begin{aligned} v^{(0,0,1)} &= P_{110}(x_1, x_2, x_3)v, \\ v(x_1, x_2, x_3 + \omega_3) &= v(x_1, x_2, x_3), \end{aligned} \tag{4}$$

$$\begin{aligned} v^{(0,1,0)} &= P_{101}(x_1, x_2, x_3)v, \\ v(x_1, x_2 + \omega_2, x_3) &= v(x_1, x_2, x_3), \end{aligned} \tag{5}$$

and

$$\begin{gathered} v^{(0,1,1)} = P_{110}(x_1, x_2, x_3)v^{(0,1,0)} + P_{101}(x_1, x_2, x_3)v^{(0,0,1)} + P_{100}v, \\ v(x_1, x_2 + \omega_2, x_3) = v(x_1, x_2, x_3), \quad v(x_1, x_2, x_3 + \omega_3) = v(x_1, x_2, x_3). \end{gathered} \tag{6}$$

Problem (4) is called an σ-*associated problem* of problem $(1), (2)$.

Problems (4), (5) and (6) are called σ-*associated problems* of problem $(1), (3)$.

Notice that:

Problem (4) is a one-dimensional periodic problem with respect to x_3 variable, depending on two parameters x_1 and x_2;

Problem (5) is a one-dimensional periodic problem with respect to x_2 variable, depending on two parameters x_1 and x_3;

Problem (6) is a two-dimensional periodic problem with respect to x_2 and x_3 variables, depending the parameter x_1.

Theorem 1. *Let problem* (4) *have only the trivial solution for every* $(x_1, x_2) \in [0, \omega_1] \times [0, \omega_2]$. *Then problem* $(1), (2)$ *has a unique solution* u *admitting the estimate*

$$\|u\|_{C^1(\Omega)} \leq M\Big(\|\varphi_1\|_{C^{1,1}(\Omega_{23})} + \|\varphi_2\|_{C^{1,1}(\Omega_{13})} + \|q\|_{C(\Omega)}\Big), \tag{7}$$

where M *is a positive number independent of* φ_1, φ_2 *and* q.

Definition 1. Problem $(1), (2)$ is called well-posed, if for every $\varphi_1 \in C^{1,1}(\Omega_{23}; \mathbb{R}^n)$, $\varphi_2 \in C^{1,1}(\Omega_{13}; \mathbb{R}^n)$ and $q \in C(\Omega; \mathbb{R}^n)$, it is uniquely solvable and its solution admits estimate (7), where M is a positive number independent of φ_1, φ_2 and q.

Theorem 2. *Let problem* $(1), (2)$ *be well-posed. Then problem* (4) *has only the trivial solution for every* $(x_1, x_2) \in [0, \omega_1] \times [0, \omega_2]$.

Corollary 1. *Let* $P_{110}(x_1, x_2, x_3) = P_{110}(x_1, x_2)$. *Then problem* $(1), (2)$ *is well-posed if and only if*

$$\det\big(I - \exp(\omega_3 P_{110}(x_1 x_2))\big) \neq 0 \ \textit{ for } \ (x_1, x_2) \in \Omega_{12}.$$

Corollary 2. *Let*

$$\widehat{P}_{110}(x_1, x_2, x_3) = \frac{1}{2}\big(P_{110}(x_1, x_2, x_3) + P^T_{110}(x_1, x_2, x_3)\big),$$

and let there exist $\sigma \in \{-1, 1\}$ $(i = 1, 2)$ *such that*

$$\sigma \int_0^{\omega_3} \widehat{P}_{110}(x_1, x_2, s)\, ds \ \textit{ is positive definite for } \ (x_1, x_2) \in \Omega_{12}.$$

Then problem $(1), (2)$ *is well-posed.*

Consider the system

$$u^{(\mathbf{1})} = P(\mathbf{x})u + q(\mathbf{x}). \tag{8}$$

By Theorem 2, problem $(8),(2)$ is *ill-posed*, since its σ-associated problem

$$v^{(0,0,1)} = 0, \quad v(x_1,x_2,x_3+\omega_3) = v(x_1,x_2,x_3)$$

has a nontrivial solution $v(x_3) \equiv 1$ for *every* $(x_1,x_2) \in [0,\omega_1] \times [0,\omega_2]$. Being ill-posed, problem $(8),(2)$ still can be uniquely solvable.

Theorem 3. *Let $P \in C^{1,1,0}(\Omega;\mathbb{R}^{n\times n})$, $q \in C^{1,1,0}(\Omega;\mathbb{R}^n)$, $\varphi_1 \in C^{2,1}(\Omega_{23})$, $\varphi_2 \in C^{2,1}(\Omega_{13})$, and let*

$$\det\bigg(\int\limits_0^{\omega_3} P(x_1,x_2,s)\,ds\bigg) \neq 0 \ \ \textit{for} \ \ (x_1,x_2) \in [0,\omega_1]\times[0,\omega_2].$$

Then problem $(8),(2)$ has a unique solution u admitting the estimate

$$\|u\|_{C^{\mathbf{1}}(\Omega)} \leq M\Big(\|\varphi_1\|_{C^{2,1}(\Omega_{23})} + \|\varphi_2\|_{C^{2,1}(\Omega_{13})} + \|q\|_{C^{1,1,0}(\Omega)}\Big),$$

where M is a positive number independent of φ_1, φ_2 and q, if and only if

$$\int\limits_0^{\omega_3} \big(P(0,x_2,s)\varphi_1(x_2,s) + q(0,x_2,s)\big)\,ds = 0 \ \ \textit{for} \ \ x_2 \in [0,\omega_2]$$

and

$$\int\limits_0^{\omega_3} \big(P(x_1,0,s)\varphi_2(x_1,s) + q(x_1,0,s)\big)\,ds = 0 \ \ \textit{for} \ \ x_1 \in [0,\omega_1].$$

Theorem 4. *Let the following conditions hold:*

(F_1) *Problem (4) has only the trivial solution for every $(x_1,x_2) \in \Omega_{12}$;*

(F_2) *Problem (5) has only the trivial solution for every $(x_1,x_3) \in \Omega_{13}$;*

(F_3) *Problem (6) has only the trivial solution for every $x_1 \in [0,\omega_1]$.*

Then problem $(1),(3)$ has a unique solution u admitting the estimate

$$\|u\|_{C^{\mathbf{1}}(\Omega)} \leq M\big(\|\varphi\|_{C^{1,1}(\Omega_{23})} + \|q\|_{C(\Omega)}\big), \tag{9}$$

where M is a positive number independent of φ and q.

Definition 2. Problem $(1),(3)$ is called well-posed, if for every $\varphi \in C^{1,1}(\Omega_{23};\mathbb{R}^n)$ and $q \in C(\Omega;\mathbb{R}^n)$, it is uniquely solvable and its solution admits estimate (9), where M is a positive number independent of φ and q.

Theorem 5. *Let problem $(1),(3)$ be well-posed. Then conditions (F_1), (F_2) and (F_3) hold.*

Corollary 3. *Let*

$$P_{110}(x_1,x_2,x_3) \equiv P_{110}(x_1),$$
$$P_{101}(x_1,x_2,x_3) \equiv P_{101}(x_1),$$
$$P_{100}(x_1,x_2,x_3) \equiv P_{100}(x_1),$$

and let

$$\det\big(I - \exp(\omega_3 P_{110}(x_1))\big) \neq 0 \ \textit{ for } \ x_1 \in [0,\omega_1],$$
$$\det\big(I - \exp(\omega_2 P_{101}(x_1))\big) \neq 0 \ \textit{ for } \ x_1 \in [0,\omega_1].$$

Then problem (1), (3) *is well-posed* ***if and only if***

$$\det\Big(P_{100}(x_1) + i\,\frac{2\pi}{\omega_3}\,mP_{110}(x_1) + i\,\frac{2\pi}{\omega_2}\,kP_{101}(x_1) + mk\,I\Big) \neq 0 \quad \textit{for} \quad x_1 \in [0,\omega_1], \quad m,k \in \mathbb{Z}.$$

Consider the equation

$$u^{(\mathbf{1})} = \sum_{\boldsymbol{\alpha}<\mathbf{1}} p_{\boldsymbol{\alpha}}(x_1,x_2)u^{(\boldsymbol{\alpha})} + q(\mathbf{x}). \tag{10}$$

Corollary 4. *Let*

$$p_{100}(x_1,x_2)\,p_{110}(x_1,x_2)\,p_{101}(x_1,x_2) < 0 \ \textit{ for } \ (x_1,x_2) \in \Omega_{12}.$$

Then problem (10), (2) *is well-posed.*

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