The Exponential Solution to Quaternion Dynamic Equations Based on a New Quaternion Hyper-Complex Space with Hyper Argument

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Abstract

In this short communication, we will introduce the notion of quaternion hyper argument to construct the non-commutative quaternion hyper argument space. By virtue of the structure of the Hilger complex plane and hyper argument space theory, we establish a theoretical framework of the quaternion hyper-complex space in which the new quaternion hyper-complex exponent, the hyper-complex logarithm are introduced. Note that the quaternion exponential functions introduced here is a solution of the linear homogeneous dynamic equation $x^{\Delta}(t) = f(t)x(t)$ under the non-commutative quaternion function f.

1 Quaternion hyper argument space and calculus

The notion of quaternion was introduced by Hamilton in 1843, which provides a type of hyper-complex numbers and extends the filed $\mathbb{C}$ of the complex numbers to a novel non-commutative division ring under the addition and multiplication operation. The study quaternion dynamic equations becomes a hot topic and some basic results were established on time scales by Wang and Agarwal et al. (see [1–6]).

In the literature [4], some important notions of the hyper-complex polar form of the quaternion numbers and a notion of the quaternion hyper argument are presented as follows.

Definition 1.1 ([4])**.** Let $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{Q}$, $\cos^{\mathbb{Q}}, \sin^{\mathbb{Q}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, we define the quaternion polar form of q by

$$q := |q| e^{\arg^{\mathbb{Q}}(q)} = |q| e^{\Theta} = |q| \big[\cos^{\mathbb{Q}} \Theta + \sin^{\mathbb{Q}} \Theta j \big],$$

where

$$\Theta = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}, \ \ \theta^{(1)}, \theta^{(2)} \in (-\pi, \pi], \ \ \theta^{(3)} \in \Big[0, \frac{\pi}{2}\Big],$$
$$\cos^{\mathbb{Q}} \Theta = \cos\theta^{(1)} \cos\theta^{(3)} + \sin\theta^{(1)} \cos\theta^{(3)} i, \quad \sin^{\mathbb{Q}} \Theta = \cos\theta^{(2)} \sin\theta^{(3)} + \sin\theta^{(2)} \sin\theta^{(3)} i,$$

and $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ satisfy the following conditions:

(i) $\cos\theta^{(1)} = \dfrac{q_0}{\sqrt{q_0^2+q_1^2}}$ and $\sin\theta^{(1)} = \dfrac{q_1}{\sqrt{q_0^2+q_1^2}}$ if $q_0+q_1i \neq 0$; $\theta^{(1)}=0$ if $q_0+q_1i=0$;

(ii) $\cos\theta^{(2)} = \dfrac{q_2}{\sqrt{q_2^2+q_3^2}}$ and $\sin\theta^{(2)} = \dfrac{q_3}{\sqrt{q_2^2+q_3^2}}$ if $q_2j+q_3k \neq 0$; $\theta^{(2)}=0$ if $q_2j+q_3k=0$;

(iii) $\cos\theta^{(3)} = \dfrac{\sqrt{q_0^2+q_1^2}}{|q|}$ and $\sin\theta^{(3)} = \dfrac{\sqrt{q_2^2+q_3^2}}{|q|}$ if $q \neq 0$; $\theta^{(1)}=\theta^{(2)}=\theta^{(3)}=0$ if $q=0$,

we call Θ the quaternion hyper-principle argument. Generally, we define the quaternion hyper argument $\operatorname{Arg}^{\mathbb{Q}}(q)$ of q satisfying

$$e^{\operatorname{Arg}^{\mathbb{Q}}(q)} := e^{\Upsilon} = \cos^{\mathbb{Q}}\Upsilon + \sin^{\mathbb{Q}}\Upsilon j,$$

where $\Upsilon = (\alpha^{(1)},\alpha^{(2)},\alpha^{(3)})^{\mathbb{Q}} \in \Gamma_q$ and

$$\Gamma_q = \Big\{\Upsilon \mid \ \cos^{\mathbb{Q}}\Theta + \sin^{\mathbb{Q}}\Theta j = \cos^{\mathbb{Q}}\Upsilon + \sin^{\mathbb{Q}}\Upsilon j\Big\}.$$

Remark 1.1. Let

$$q = |q|e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}}, \quad p = |p|e^{(\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})^{\mathbb{Q}}},$$

then

$$\arg^{\mathbb{Q}}(qp) \neq \big(\theta^{(1)}+\gamma^{(1)},\theta^{(2)}+\gamma^{(2)},\theta^{(3)}+\gamma^{(3)}\big)^{\mathbb{Q}}$$

in general.

Remark 1.2. Let

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}, \quad \arg^{\mathbb{Q}}(\overline{q}) = (\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})^{\mathbb{Q}},$$

then

$$\theta^{(1)}+\gamma^{(1)}=0, \quad |\theta^{(2)}-\gamma^{(2)}|=\pi \ \text{ and } \ \theta^{(3)}=\gamma^{(3)}.$$

Remark 1.3. Note that the quaternion hyper-principle argument

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}$$

is unique for each fixed q.

Remark 1.4. Let

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}} \ \text{ and } \ \operatorname{Arg}^{\mathbb{Q}}(q) = (\alpha^{(1)},\alpha^{(2)},\alpha^{(3)})^{\mathbb{Q}}, \ \ n_1,n_2,n_3 \in \mathbb{Z},$$

then

$$\alpha^{(1)} = \theta^{(1)}+2n_1\pi, \quad \alpha^{(2)} = \theta^{(2)}+2n_2\pi, \quad \alpha^{(3)} = \theta^{(3)}+2n_3\pi,$$

or

$$\alpha^{(1)} = \theta^{(1)}+2n_1\pi, \quad \alpha^{(2)} = \theta^{(2)}+2n_2\pi+\pi, \quad \alpha^{(3)} = -\theta^{(3)}+2n_3\pi,$$

or

$$\alpha^{(1)} = \theta^{(1)}+2n_1\pi+\pi, \quad \alpha^{(2)} = \theta^{(2)}+2n_2\pi+\pi, \quad \alpha^{(3)} = \theta^{(3)}+2n_3\pi+\pi,$$

etc., this indicates that Γ_q is an infinite set.

Remark 1.5. Note that

$$\Big\{q \mid \ q \in \mathbb{Q},\ \arg^{\mathbb{Q}}(q) = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}},\ \theta^{(3)} = 0\Big\} = \mathbb{C}$$

and

$$\Big\{q \mid \ q \in \mathbb{Q},\ \arg^{\mathbb{Q}}(q) = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}},\ \theta^{(3)} = 0,\ \theta^{(1)} = 0 \text{ or } \pi\Big\} = \mathbb{R}.$$

Moreover,

$$\begin{aligned} e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}} &= e^{\theta^{(1)}i} \ \text{ if } \ \theta^{(3)} = 0; \\ q \in \mathbb{R} \ \text{ and } \ e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}} &= 1 \ \text{ if } \ \theta^{(3)} = \theta^{(1)} = 0; \\ e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}} &= e^{\theta^{(2)}i}j \ \text{ if } \ \theta^{(3)} = \frac{\pi}{2}\,. \end{aligned}$$

Remark 1.6. Note that for $\arg^{\mathbb{Q}}(q) = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}}$, it follows that

$$q = a + bj = |a|e^{\theta^{(1)}i} + |b|e^{\theta^{(2)}i}j = |q|\big(e^{\theta^{(1)}i}\cos\theta^{(3)} + e^{\theta^{(2)}i}\sin\theta^{(3)}j\big),$$

where $a, b \in \mathbb{C}$.

2 The quaternion hyper-complex space

Definition 2.1 ([4])**.** Let $h > 0$, $\mathbb{Q} = \mathbb{C}_1 \times \mathbb{C}_2$, $q = (q_0 + q_1 i) + (q_2 + q_3 i)j \in \mathbb{Q}$, $q_0 + q_1 i \in \mathbb{C}_1$ and $q_2 + q_3 i \in \mathbb{C}_2$. Then $\mathbb{C}_1$ is called the sub-complex plane of the quaternion hyper-complex space, and $\mathbb{C}_2$ is called the imaginary-complex plane of the quaternion hyper-complex space. Moreover, we define the Hilger quaternion number set as

$$\mathbb{Q}_h := \Big\{q \in \mathbb{Q}:\ q \neq -\frac{1}{h}\Big\}.$$

Let $q = a + bj \in \mathbb{Q}_h,\ a, b \in \mathbb{C},\ \theta^{(1)} = \operatorname{Im}_h(a),\ \theta^{(2)} = \operatorname{Im}_h(b),\ \theta^{(3)} = \operatorname{Im}_h(|a| + |b|j)$, then the schematic diagram of the quaternion hyper-complex space is showed by **Figure** 1. For $h = 0$, then $\mathbb{Q}_0 = \mathbb{Q}$.

Now, let

$$\chi_h(q) = \begin{cases} \dfrac{\ln|1 + hq|}{h} & \text{for } \ h > 0, \\ q_0 & \text{for } \ h = 0, \end{cases} \qquad \mathbb{A}_h(q) = \begin{cases} \dfrac{1}{h} \cdot \arg^{\mathbb{Q}}(1 + hq) & \text{for } \ h > 0, \\ \lim\limits_{h\to} \dfrac{1}{h} \cdot \arg^{\mathbb{Q}}(1 + hq) & \text{for } \ h = 0, \end{cases}$$

we introduce the hyper-complex cylinder transformation $\xi_h^{\mathbb{Q}} : \mathbb{Q}_h \to \mathbb{Z}_h^{\mathbb{Q}}$ by

$$\xi_h^{\mathbb{Q}}(q) = \chi_h(q) + \mathbb{A}_h(q) = \begin{cases} \dfrac{\ln|1 + hq|}{h} + \dfrac{1}{h} \cdot \arg^{\mathbb{Q}}(1 + hq) & \text{for } \ h > 0, \\ q_0 + \lim\limits_{h\to 0} \dfrac{1}{h} \cdot \arg^{\mathbb{Q}}(1 + hq) & \text{for } \ h = 0, \end{cases}$$

where

$$\mathbb{Z}_h^{\mathbb{Q}} = \Big\{q \in \mathbb{Q}:\ \theta^{(1)}, \theta^{(2)} \in \Big(-\frac{\pi}{h}, \frac{\pi}{h}\Big], \theta^{(3)} \in \Big[0, \frac{\pi}{2}\Big]\Big\}.$$

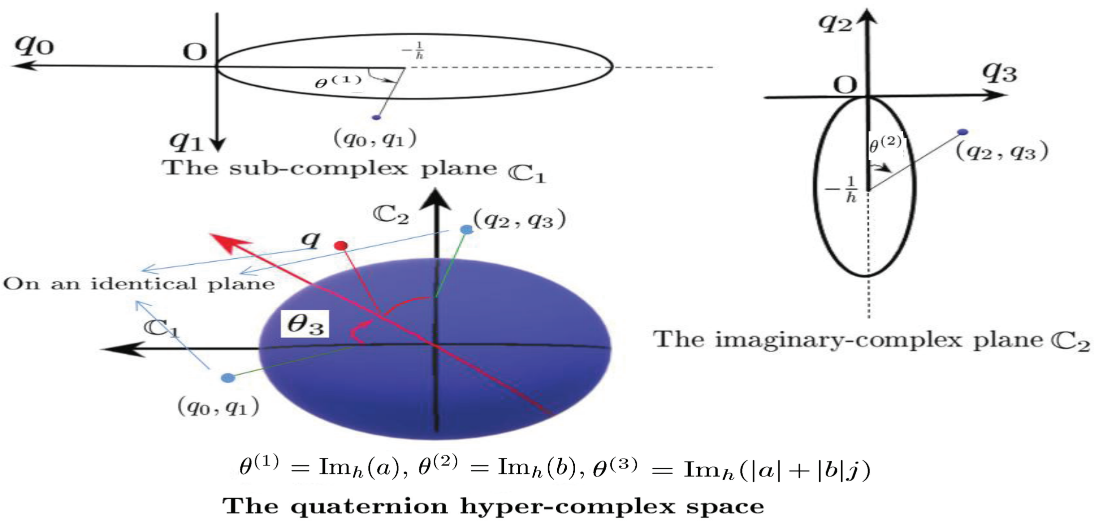

Figure 1. The geometric diagram of the quaternion hyper-complex space.

Remark 2.1. Let $h > 0$, the Hilger complex numbers $\mathbb{C}_h = \{z \in \mathbb{C} \mid \ z \neq -\frac{1}{h}\}$, then $\mathbb{C}_h \subset \mathbb{Q}_h$. In fact, let $p, q \in \mathbb{Q}_h$ and $\arg^{\mathbb{Q}}(q) = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}}$, $\arg^{\mathbb{Q}}(p) = (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})^{\mathbb{Q}}$, we have

$$\arg^{\mathbb{Q}}(q) \oplus_{\mathbb{Q}} \arg^{\mathbb{Q}}(p) = \theta^{(1)} i + \gamma^{(1)} i,$$
$$\arg^{\mathbb{Q}}(q) \ominus_{\mathbb{Q}} \arg^{\mathbb{Q}}(p) = \theta^{(1)} i - \gamma^{(1)} i,$$
$$b \cdot \arg^{\mathbb{Q}}(q) = b\theta^{(1)} i,$$

where $b \in \mathbb{R}$ and $\theta^{(2)} = \theta^{(3)} = \gamma^{(2)} = \gamma^{(3)} = 0$, it means that the operations $\oplus_{\mathbb{Q}}$ and $\ominus_{\mathbb{Q}}$ will turn into the classical operations $+$ and $-$ when $\theta^{(2)} = \theta^{(3)} = \gamma^{(2)} = \gamma^{(3)} = 0$, by Remark 1.5, we can obtain $\mathbb{C}_h \subset \mathbb{Q}_h$.

Next, we will introduce the quaternion hyper-complex logarithm in the quaternion hyper-complex space.

Definition 2.2 ([4])**.** Let $q \in \mathbb{Q}$, $q \neq 0$. We define the quaternion hyper-complex logarithm by

$$\mathrm{Log}^{\mathbb{Q}}(q) := \ln|q| + \arg^{\mathbb{Q}}(q).$$

Remark 2.2. Note that $e^{\mathrm{Log}^{\mathbb{Q}}(q)} = q$ for any nonzero quaternion number $q \in \mathbb{Q}$. In fact,

$$e^{\mathrm{Log}^{\mathbb{Q}}(q)} = e^{\ln|q| + \arg^{\mathbb{Q}}(q)} = e^{\ln|q|} e^{\arg^{\mathbb{Q}}(q)} = |q| e^{\arg^{\mathbb{Q}}(q)} = q.$$

Remark 2.3. Let $q, p \in \mathbb{Q}$,

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}} \ \text{ and } \ \arg^{\mathbb{Q}}(p) = (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})^{\mathbb{Q}},$$

then

$$\mathrm{Log}^{\mathbb{Q}}(qp) = \ln|q| + \ln|p| + (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}} \oplus_{\mathbb{Q}} (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})^{\mathbb{Q}}.$$

3 The quaternion hyper-complex exponential function and dynamic equation on time scales

Definition 3.1 ([4])**.** Let $t,s\in\mathbb{T}$, $f:\mathbb{T}\to\mathbb{Q}$, $1+\mu(t)f(t)\neq 0$ for any $t\in\mathbb{T}^\kappa$, then we define $\widehat{x}(t,t_0)$ and $\widetilde{x}(t,t_0)$ as follows:

(i) $\widehat{x}(t,s):=e^{\int\limits_s^t \frac{\ln|1+\mu(\tau)f(\tau)|}{\mu(\tau)}\,\Delta\tau+\int\limits_s^t \frac{1}{\mu(\tau)}\cdot\arg^{\mathbb{Q}}(1+\mu(\tau)f(\tau))\,\Delta\tau}$ if $\mu(\tau)>0$ for any $\tau\in[s,t]_{\mathbb{T}}$.

(ii) If $\lim\limits_{u\to 0}\frac{1}{u}\cdot\arg^{\mathbb{Q}}(1+uf(t))=\Theta(t)$ and $\Theta(t)$ is an integrable quaternion hyper argument function, then we define

$$\widetilde{x}(t,s):=e^{\int\limits_s^t f_0(\tau)\,\mathrm{d}\tau+\int\limits_s^t \Theta(\tau)\,\mathrm{d}\tau}$$

if $\mu(\tau)=0$ for any $\tau\in[s,t]_{\mathbb{T}}$, where $f(t)=f_0(t)+f_1(t)i+f_2(t)j+f_3(t)k$.

Generally, Based on the hyper-complex cylinder transformation $\xi^{\mathbb{Q}}_{\mu(t)}:\mathbb{Q}_h\to\mathbb{Z}^{\mathbb{Q}}_h$ by

$$\begin{aligned}\xi^{\mathbb{Q}}_{\mu(t)}(f(t))&=\chi_{\mu(t)}(f(t))+\mathbb{A}_{\mu(t)}(f(t))\\&=\begin{cases}\dfrac{\ln|1+\mu(t)f(t)|}{\mu(t)}+\dfrac{1}{\mu(t)}\cdot\arg^{\mathbb{Q}}(1+\mu(t)f(t)) & \text{for } \mu(t)>0,\\ f_0(t)+\Theta(t) & \text{for } \mu(t)=0,\end{cases}\end{aligned}$$

we define the quaternion hyper-complex exponential function by

$$e^{\mathbb{Q}}_f(t,s):=e^{\int\limits_s^t \xi^{\mathbb{Q}}_{\mu(\tau)}(f(\tau))\,\Delta\tau}=e^{\int\limits_s^t \chi_{\mu(\tau)}(f(\tau))\,\Delta\tau+\int\limits_s^t \mathbb{A}_{\mu(\tau)}(f(\tau))\,\Delta\tau}.$$

The following result is valid.

Theorem 3.1 ([4])**.** *Let $s,r,t\in\mathbb{T}$, $f:\mathbb{T}\to\mathbb{Q}$, $1+\mu(t)f(t)\neq 0$ for any $t\in\mathbb{T}^\kappa$. Then*

(i) $e^{\mathbb{Q}}_f(s,s)=1$;

(ii) $e^{\mathbb{Q}}_f(t,r)e^{\mathbb{Q}}_f(r,s)=e^{\mathbb{Q}}_f(t,s)$;

(iii) $(e^{\mathbb{Q}}_f(t,s))^\Delta=f(t)e^{\mathbb{Q}}_f(t,s)$;

(iv) $(e^{\mathbb{Q}}_f(s,t))^\Delta=e^{\mathbb{Q}}_f(s,t)(1+\mu(t)f(t))^{-1}[-f(t)]$ *if t is a right scattered point on $\mathbb{T}$.*

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