The Exact Baire Class of the Asymptotic ε-Capacity of a Family of Non-Autonomous Dynamical Systems Continuously Depending on a Parameter

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In this paper we consider a parametric family of non-autonomous dynamical systems defined on a compact metric space and continuously depending on a parameter from some topological space. For any such family, we study the asymptotic ε-capacity of its dynamical systems as a function of the parameter from the standpoint of the Baire classification.

As a measure of "massiveness" of a compact metric space (X,d), A. N. Kolmogorov in the paper [1] introduced the notion of ε-capacity which is defined as the maximum number of ε-distinguishable elements in X. Using this notion, we give the definition of the topological entropy of a non-autonomous dynamical system [2].

Let $\mathcal{F}\equiv(f_1,f_2,\dots)$ be a sequence of continuous mappings from X to X. For any positive integer n, denote by F_n the subsequence $(f_n,f_{n+1},\dots)$ of the sequence $\mathcal{F}$. Along with the original metric d we define on X an additional system of metrics

$$d_k^{F_n}(x,y)=\max_{0\le i\le k-1} d(f_n^{\circ i}(x),f_n^{\circ i}(y)),$$
$$\big(f_n^{\circ i}\equiv f_{n+(i-1)}\circ\dots\circ f_n,\ f_n^{\circ 0}\equiv \mathrm{id}_X\big),\quad x,y\in X,\quad k,n\in\mathbb{N}.$$

For any $k\in\mathbb{N}$ and $\varepsilon>0$, we denote by $N_d(F_n,\varepsilon,k)$ the maximum number of points in X such that their pairwise $d_k^{F_n}$-distances are greater than ε. Such a set of points will be called (F_n,ε,k)-separated. Then the *ε-capacity* $h_d(\mathcal{F},\varepsilon)$ and *asymptotic ε-capacity* $h_d^*(\mathcal{F},\varepsilon)$ of the non-autonomous dynamical system $(X,\,\mathcal{F})$ are defined by the equalities

$$h_d(\mathcal{F},\varepsilon)=\varlimsup_{k\to\infty}\frac{1}{k}\,\ln N_d(F_1,\varepsilon,k), \tag{1}$$
$$h_d^*(\mathcal{F},\varepsilon)=\sup_{n\in\mathbb{N}}\lim_{\varepsilon\to 0}\limsup_{k\to\infty}\frac{1}{k}\,\ln N_d(F_n,\varepsilon,k). \tag{2}$$

It follows directly from formulas (1) and (2) that

$$h_d(\mathcal{F},\varepsilon)\le h_d^*(\mathcal{F},\varepsilon)$$

holds for any sequence $\mathcal{F}$. As the following example shows, quantities (1) and (2) may not coincide. Let us equip the set Ω_2 of two-sided sequences

$$x=(\dots,x_{-2},x_{-1},x_0,x_1,x_2,\dots),\ x_k\in\{0,\,1\},$$

with the metric

$$d_{\Omega_2}(x,y)=\begin{cases}0 & \text{if } x=y;\\ 2^{-\min\{|i|:\,x_i\ne y_i\}} & \text{if } x\ne y.\end{cases}$$

Note that the resulting metric space (Ω_2, d_{Ω_2}) is compact and homeomorphic to the Cantor set on the segment $[0,1]$ with the metric induced by the standard metric of the real line. Let $\sigma:\Omega_2\to\Omega_2$ stand for the left shift by one element:

$$\sigma((\dots,x_{-1},x_0,x_1,\dots))=(\dots,x_0,x_1,x_2,\dots),$$

and $\chi:\Omega_2\to\Omega_2$ be the map that takes any element from Ω_2 to the sequence of zeros:

$$\chi((\dots,x_{-1},x_0,x_1,\dots))=(\dots,0,0,0,\dots).$$

Then for the sequence $\mathcal{F}\equiv(\chi,\sigma,\sigma,\dots)$ and $\varepsilon<1/2$ we have

$$h_d(\mathcal{F},\varepsilon)=0<\ln 2=h_d^*(\mathcal{F},\varepsilon).$$

Note that in this example the equality

$$h_d^*(\mathcal{F},\varepsilon)=\limsup_{k\to\infty}\frac{1}{k}\,\ln N_d(F_2,\varepsilon,k)$$

holds from which we obtain

$$h_d^*(\mathcal{F},\varepsilon)=\max_{n\in\mathbb{N}}\limsup_{k\to\infty}\frac{1}{k}\,\ln N_d(F_n,\varepsilon,k).$$

In the general case, as the following example shows, the supremum over n in formula (2) cannot be replaced by the maximum.

Let Λ_2 be the set of infinite matrices of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots\\ a_{21} & a_{22} & a_{23} & \dots\\ a_{31} & a_{32} & a_{33} & \dots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_{ij}\in\{0,1\}$; on this set we introduce the metric

$$d_{\Lambda_2}(A,B)=\begin{cases} 0 & \text{if } A=B;\\ 2^{-\min\{\max\{i,j\}:\,a_{ij}\neq b_{ij}\}} & \text{if } A\neq B.\end{cases}$$

Consider the sequence $\mathcal{F}\equiv(f_1,f_2,\dots)$ of continuous mappings $\Lambda_2\to\Lambda_2$ defined by

$$f_1\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots\\ a_{21} & a_{22} & a_{23} & \dots\\ a_{31} & a_{32} & a_{33} & \dots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right)=\begin{pmatrix} a_{11+1} & a_{12+1} & a_{13+1} & \dots\\ 0 & 0 & 0 & \dots\\ 0 & 0 & 0 & \dots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$f_2\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots\\ a_{21} & a_{22} & a_{23} & \dots\\ a_{31} & a_{32} & a_{33} & \dots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right)=\begin{pmatrix} a_{11+1} & a_{12+1} & a_{13+1} & \dots\\ a_{21+2} & a_{22+2} & a_{23+2} & \dots\\ 0 & 0 & 0 & \dots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},\ \dots .$$

It follows that for all $n\in\mathbb{N}$ and $\varepsilon<1/2$, the inequality

$$+\infty=h_d^*(\mathcal{F})>\max_{1\le m\le n}\overline{\lim_{k\to\infty}}\,\frac{1}{k}\,\ln N_d(F_n,\varepsilon,k)$$

holds.

For a given metric space $\mathcal{M}$ and a sequence of continuous mappings

$$\mathcal{F} \equiv (f_1, f_2, \dots), \quad f_i : \mathcal{M} \times X \to X, \tag{3}$$

we form the functions

$$\mu \longmapsto h_d\big(\mathcal{F}(\mu, \cdot\,), \varepsilon\big), \tag{4}$$

$$\mu \longmapsto h_d^*\big(\mathcal{F}(\mu, \cdot\,), \varepsilon\big). \tag{5}$$

It was proved in [3] that for any metric space $\mathcal{M}$, compact metric space X and sequence of mappings (3) function (4) belongs to the second Baire class and in general does not belong to the first Baire class. Recall that the zeroth Baire class on a topological space $\mathcal{M}$ consists of all continuous functions, and for any positive integer p functions of the p-th Baire class are the functions that are pointwise limits of sequences of functions belonging to the $(p-1)$-th class.

In the same paper [3] it was proved for a complete metric space $\mathcal{M}$ that the set of points of upper semicontinuity of function (4) is an everywhere dense G_δ-set.

In this paper similar results are obtained for function (5).

Theorem 1. *For any sequence of mappings* (3)*, function* (5) *belongs to the second Baire class. Furthermore, its set of points of upper (lower) semicontinuity is a G_δ-set (an $F_{\sigma\delta}$-set).*

Theorem 2. *If $\mathcal{M} = X = \Omega_2$, then there exists a sequence of mappings* (3) *such that for any $\varepsilon \in (0; 1/4]$ function* (5) *does not belong to the first Baire class on the space $\mathcal{M}$.*

Theorem 3. *If a space $\mathcal{M}$ is complete, then the set of points of upper semicontinuity of function* (5) *is an everywhere dense G_δ-set.*

References

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